Mathematical Reviews

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HISTORY

✶Selections Illustrating the History of Greek Mathematics. With an English translation by Ivor Thomas. I. From Thales to Euclid. The Loeb Classical Library. Harvard University Press, Cambridge, Mass., 1939. xvi+505 pp.

This book contains a collection of Greek mathematical texts in Greek with English translations and short explanatory notes. It belongs formally to the time from the beginnings of Greek mathematics to Euclid (included) but reflects very clearly by its real content the fact that we really do not know anything about the periods before the Academy. It is therefore simply misleading when topics treated by Euclid or even in much later times are classified under "Pythagorean Arithmetic" (as for instance page 66 ff., quotations from Euclid VII, Philolaos and Nicomachus) or "Pythagorean Geometry." This criticism of the general historical tendency of the arrangement and interpretation of the material does not affect the usefulness of this valuable selection. *O. Neugebauer* (Providence, R. I.).

Rome, A. Le problème de l'équation du temps chez Ptolémée. Ann. Soc. Sci. Bruxelles. Sér. I. **59**, 211–224 (1939). [MF 110]

This paper is based on the unpublished commentary of Theon on Ptolemy's Almagest, Book III, on the "great commentary" on the "easy tables" (also unpublished), and on the "little commentary" on the same work (published by Halma in 1822). From these sources it is possible to conclude that the tables on rectascension and on obliquascension, given by Theon in his "great commentary," represent the original Ptolemaic form. Furthermore it is shown that from the position of the sun at the moment chosen by Ptolemy in the "easy tables" as the point $t=0$, it follows that the "time equation" contains positive numbers only. In the same way the choice of the epoch made by Albattani results in a "time equation" with only negative numbers. The paper concludes with remarks on perhaps analogous tables of Serapion, who lived before Ptolemy.
O. Neugebauer (Providence, R. I.).

Loria, Gino. Un problema aritmetico che può essere stato risolto dagli antichi Greci. An. Fac. Ci. Pôrto **24**, 3–7 (1939). [MF 557]

It is shown that from the "side" and "diagonal-numbers" which appear in Theon Smyrn. [ed. Hiller, pp. 43–44] and which are solutions of $2u^2\pm1=v^2$ can be deduced those solutions of $x^2+y^2=z^2$ in integers, where $z-y=1$.
O. Neugebauer (Providence, R. I.).

✶Michel, Henri. Introduction à l'étude d'une collection d'instruments anciens de mathématiques. De Sikkel, Anvers, 1939. 105 pp.

Bompiani, Enrico. Alcune costruzioni di coniche. Boll. Un. Mat. Ital. **1**, 372–373 (1939). [MF 580]

Si ricorda una costruzione di Guidubaldo del Monte per l'ellisse e se ne dà una analoga per l'iperbole. *Summary.*

Dehn, M. and Hellinger, E. On James Gregory's Vera Quadratura. The James Gregory Tercentenary Memorial Volume, 468–478 (1939). [MF 267]

Appreciation of the "Vera circuli et hyperbolae quadratura" of James Gregory, published at Padua in 1667. Gregory's treatise contains important results, and perhaps more profound ideas. These include the simultaneous analytical generation of circular and hyperbolic functions, the idea of the general, infinite double sequence generated by iteration:

$$a_{n+1}=\varphi(a_n, b_n),\quad b_{n+1}=\psi(a_n, b_n),\quad |b_{n+1}-a_{n+1}|<|b_n-a_n|,$$

in the discussion of which the term "convergent" appears, apparently for the first time, and an example of which leads closely to the theory of the arithmetical-geometrical mean. Gregory had some ideas on general functional dependency and endeavored to prove the impossibility of the solution of certain problems by means of given operations (for example, the squaring of the circle). *D. J. Struik.*

Hofmann, Jos. E. Weiterbildung der logarithmischen Reihe Mercators in England. II. Deutsche Math. **4**, 556–562 (1939). [MF 136]

This is a carefully compiled and well documented account of the work of Newton and of James Gregory on the logarithmic series (first published in 1668 by Mercator) for the period 1665–1675. The author assesses the value of their discoveries on the basis of their work already published. A more accurate estimate, he says, could be made when further original material is published, particularly the letters of Collins and notes of Gregory discovered lately in St. Andrews. [The Gregory Memorial Volume (Royal Society of Edinburgh) will contain full details. It will be published in October, 1939.] Newton's investigation anticipated and Gregory's followed that of Mercator. *H. W. Turnbull.*

Pelseneer, Jean. Lettres inédites de Newton. Osiris **7**, 523–555 (1939). [MF 426]

Bortolotti, Ettore. I primi algoritmi infiniti nelle opere dei matematici italiani del secolo XVII. Boll. Un. Mat. Ital. **1**, 351–371 (1939). [MF 579]

Frère Namase-Marie (J. M. Oudin). Sur la détermination de la date de Pâques. Démonstration générale de la formule de Gauss. Ann. Soc. Sci. Bruxelles. Sér. I. **59**, 225–256 (1939). [MF 111]

Saltykow, M. N. L'Oeuvre de Jacobi dans le domaine des équations aux dérivées partielles du premier ordre. Bull. Sci. Math. **63**, 213–228 (1939). [MF 366]

Turnbull, H. W. James Gregory (1638–1675). University of St. Andrews James Gregory Tercentenary, St. Andrews, 1939, 5–11. [MF 82]

Schnirelman, L. G. (1905–1938). Uspekhi Matem. Nauk **6**, 3–8 (1939). (Russian) [MF 394]

Archibald, R. C. Gino Loria. Osiris 7, 5–30 (1939). [MF 423]

Zassenhaus, Hans. Zum Gedenken an Hans Fitting. Jber. Deutsch. Math. Verein. **49**, 93–96 (1939). [MF 677]

FOUNDATIONS OF ANALYSIS

Tarski, Alfred. On undecidable statements in enlarged systems of logic and the concept of truth. J. Symbolic Logic **4**, 105–112 (1939). [MF 463]

Suppose L is a formalized logical system which, like Principia Mathematica, suffices for the construction of elementary number theory. Let E be any class of statements in L such that the definition of E can be formalized in L, and let $z_{(E)}$ be the formalization in L of the statement that z is a member of E. Using a construction of Gödel, the author has shown [Studia Philos. **1**, 261–405 (1935)] that there exists in L a statement z such that it is demonstrable that *not*-z is equivalent to $z_{(E)}$. Roughly speaking, z is a statement which says concerning itself "I am not a member of E."

In the present paper the author uses this construction to prove the existence for classes E of various kinds, of statements undecidable in E, that is, statements z such that neither z nor *not*-z is a member of E. After defining content-consistency, he proves that, for any (definable) content-consistent enlargement of the class D of demonstrable statements of L, there exists a statement which is undecidable in the class. In the paper referred to above, the author has shown how it is possible, using means not formalizable in L, to determine what are the "true" statements of L. He now shows that for any (definable) class E of true statements of L there exists a statement which is undecidable in E. He also shows that any class E of true statements is content-consistent. It is shown that the undecidable statements which he constructs are all true, though they may not be demonstrable in L. For a system L which is based on the theory of types, the proof of the following result is outlined: If there exist statements in L of a given type $\nu>1$, then there exists in L a statement of type ν which is not demonstrably equivalent to any statement in L of type less than ν. It follows that there are statements from the theory of real numbers which are not equivalent to any statement of elementary number theory, and statements from function theory which are not equivalent to any statement of the theory of real numbers. *O. Frink.*

Mostowski, Andrzej. Bemerkungen zum Begriff der inhaltlichen Widerspruchsfreiheit. J. Symbolic Logic **4**, 113–114 (1939). [MF 464]

A. Tarski [cf. the foregoing review] has given a definition of the content-consistency of a class E of statements of a formalized logical system, emphasizing that the concept is relative to a particular definition of the class E. In the present paper it is shown that a class E which is an enlargement of the class of all demonstrable statements of the system is content-consistent with respect to every one of its definitions if and only if it consists entirely of true statements. This proves the existence of classes which are content-consistent with respect to one definition but not another. It is also shown that every class of statements which is consistent, and also recursive in the sense of Gödel and Herbrand, is content-consistent with respect to at least one of its definitions. It is shown that there exist definable classes of statements which are consistent, but are not content-consistent with respect to any definition. An example is the class defined by Lindenbaum which is both consistent and complete. *O. Frink* (State College, Pa.).

Teichmüller, Oswald. Braucht der Algebraiker das Auswahlaxiom? Deutsche Math. **4**, 567–577 (1939). [MF 525]

Teichmüller discusses the possibility of avoiding the Axiom of Choice (Principle A) or the equivalent Theorem of Well Ordering (Principle B) in algebra by substituting for them two principles deducible from them: Principle C: Let $\mathfrak{M}$ be a set. To every $x \epsilon \mathfrak{M}$ and every natural number n let there correspond a non-void subset $\mathfrak{N}_n(x)$ of $\mathfrak{M}$. Let $x_1 \epsilon \mathfrak{M}$ be given. Then there exists a sequence $x_1, x_2, \cdots$ with $x_{n+1} \epsilon \mathfrak{N}_n(x_n)$. Principle D: Let there be given a set $\mathfrak{M}$ and a set of propositional functions $\mathfrak{A}(x_1, \cdots, x_n)$, such that each $\mathfrak{A}$ has a meaning (either true or false) when any $n \geqq 1$ elements of $\mathfrak{M}$ are used as arguments. Then there is a maximal (possibly void) subset $\mathfrak{T}$ of $\mathfrak{M}$ such that all $\mathfrak{A}$'s are true for arbitrary $x_1, \cdots, x_n \epsilon \mathfrak{T}$. He gives two other formulations of Principle D which do not involve propositional functions.

He asserts and indicates by examples on ideals, dependence, algebraic closure, and other topics, that the algebraist needs only C and D for his proofs rather than A or B. Finally he shows that A is a consequence of D, but that D is weaker in that to prove A for a set of cardinal g he must assume D for a set of cardinal 2^g. *M. Hall.*

Nagel, Ernest. The formation of modern conceptions of formal logic in the development of geometry. Osiris 7, 142–224 (1939). [MF 425]

This paper offers a detailed historical and critical exposition of some major developments in pure geometry during the 19th century and their incidence upon some central ideas of modern logic. It examines with considerable fullness the rise of the conception of geometry as a non-quantitative science of extension, and the subsequent retreat from that view due to maturing appreciation of postulational method. The steps in the formalization of geometry and the significance of the growth of demonstrative geometry in the direction of abstractness and generality are discussed in terms comprehensible to all. The closing discussion of recent mathematical views of the nature of geometry is familiar to all mathematicians interested in logical theory but should prove of service to logicians unequipped with intensive technical training in mathematics. The historical material (whose isolated items are readily available to the bibliographer) is to be appreciated for the continuity and accuracy of its account. *A. A. Bennett* (Providence, R. I.).

Lenzen, V. F. Physical geometry. Amer. Math. Monthly **46**, 324–334 (1939). [MF 156]

Mimura, Yositaka and Hosokawa, Tôyomon. Space, time and laws of nature. J. Sci. Hirosima Univ. Ser. A **9**, 217–225 (1939). [MF 673]

✱**Locher-Ernst, Louis. Geometrisieren im Bereiche wichtigster Kurvenformen.** Orell Füssli, Zürich, 1939. 64 pp. RM 2.90.

This booklet is composed of six lectures that the author gave to a mixed group of scientists, artists, physicians, and mathematical amateurs. The lectures pertain to philosophy

rather than mathematics. They are concerned, for the most part, with illustrating and implementing some views of the German mystic Rudolf Steiner regarding the nature of mathematical thought, the origins of geometry, and similar speculations. To this end circles, ellipses, hyperbolas, Cassinian ovals, spirals, hypocycloids and epicycloids are exploited, though (it seems to the reviewer) without success.

L. M. Blumenthal (Columbus, Mo.).

ALGEBRA

Linear Algebra, Polynomials, Invariants

✶Aitken, A. C. Determinants and Matrices. Oliver and Boyd, Edinburgh, 1939. vii+135 pp. 4/6.

This treatise begins from elementary considerations and develops the theory and notation of matrices and determinants alternately in such a way that each supports and simplifies the other. The general properties of determinants and matrices are covered, and there is a full treatment of rank and of simultaneous equations. A chapter is devoted to important special types, such as alternants, circulants, bigradients, Jacobians, Hessians, Wronskians and others. There are numerous illustrative examples.

Blumenthal, Leonard M. Metric methods in determinant theory. Amer. J. Math. **61**, 912–922 (1939). [MF 286]

Let $S_{n,r}$ denote the n-dimensional spherical shell of radius r metrized with the "shorter arc" distance. Let Σ be an abstract set of m elements $p, q, \cdots$, in which a distance pq is defined subject to the conditions: $pq=qp\geqq 0$, $pq=0$ if and only if $p=q$. Such a set Σ with $m\geqq n+3$ is called a pseudo-$S_{n,r}$ set, if (i) any subset of Σ of $n+2$ elements is congruently imbeddable in $S_{n,r}$, (ii) the whole set Σ can not be congruently imbedded in $S_{n,r}$. The problem of describing all such pseudo-$S_{n,r}$ sets (with the additional condition that $p, q\epsilon\Sigma$ imply $pq<\pi r$) was recently solved by Blumenthal and Thurman [Proc. Nat. Acad. Sci. U. S. A. **24**, 557–558 (1938)]. The results are strikingly simple and are here applied to determinant theory. One of the four determinant theorems thus established may be stated as follows: Let $\Delta=|r_{ij}|$, $r_{ij}=r_{ji}$, $-1<r_{ij}<1$ $(i\neq j)$, $r_{ii}=1$ $(i, j=1, \cdots, m)$, $m>n+3$. If (i) every principal minor of order less than $n+2$ is not less than 0, (ii) every principal minor of order $n+2$ vanishes, (iii) at least one principal minor of order $n+3$ is not equal to 0, then upon multiplication of appropriate rows and the same numbered columns of Δ by -1, each element outside the principal diagonal has the value $-1/(n+1)$. The author concludes with some very suggestive unsolved maximum-minimum problems concerning distances among a finite set of points of $S_{n,r}$, for the statements of which the reader is referred to the original paper.

I. J. Schoenberg (Waterville, Me.).

Jacobson, N. An application of E. H. Moore's determinant of a Hermitian matrix. Bull. Amer. Math. Soc. **45**, 745–748 (1939). [MF 337]

Let Φ be a ring, with unit element 1, which has a unique element $\frac{1}{2}$ such that $\frac{1}{2}+\frac{1}{2}=1$, and in which there is defined an involution $a\to\bar{a}$ such that the subring Σ of symmetric elements $(a=\bar{a})$ is contained in the center of Φ. The author shows that E. H. Moore's [Moore and Barnard: General Analysis I, Amer. Phil. Soc. Publication, chap. 2] definition of the determinant of a Hermitian matrix can be applied to a matrix A of order n, with elements in Φ, which is Hermitian with respect to the involution in Φ. It follows from some of Moore's results that such a matrix A satisfies an equation of degree n with coefficients in Σ that are polynomials in the elements of the matrix A. This result is applied to show that, if B is a matrix of $2n$ rows and columns with elements in a field of characteristic not equal to 2 such that $R^{-1}B'R=B$, where R is any nonsingular skew symmetric matrix, then the characteristic polynomial $f(x)$ of B has the form $[\varphi(x)]^2$, where the coefficients of $\varphi(x)$ are polynomials in the elements of B and $\varphi(B)=0$. If the elements of the general matrix B are regarded as indeterminates, then $\varphi(x)$ is irreducible.

N. H. McCoy.

Williamson, John. The exponential representation of canonical matrices. Amer. J. Math. **61**, 897–911 (1939). [MF 285]

A matrix C with real or complex elements is called canonical if

$$CGC'=G,\quad G=\begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix},$$

where E is the unit matrix and C' is the transposed of C. It was proved by Wintner [Ann. Mat. Pura Appl. (4) **13**, 105 (1934)] that $C=\exp(GS)$ is canonical for every symmetric matrix S. It is now shown that a canonical matrix C over the complex field can be represented in exponential form if and only if, among the elementary divisors of C, none of the form $(\lambda+1)^{2r}$ occurs an odd number of times. However, every C which cannot be so represented is the limit of canonical matrices which can be so represented. A real C has a real exponential representation if and only if every real elementary divisor of the form $(\lambda-a)^r$, $a<0$, occurs an even number of times and, if $(\lambda+1)^{2k}$ occurs $2m$ times, the index associated with this elementary divisor is even. A real C has a real exponential representation, or is the limit of C's which have, if and only if no negative number appears an odd number of times among the latent roots of C. Whether the field is real or complex, a C which has no exponential representation is the product of two canonical matrices, one of period two, both of which have exponential representations. Examples are given to show the existence of the exceptional types.

C. C. MacDuffee (Madison, Wis.).

Chanler, Josephine H. The invariant theory of the ternary trilinear form. Duke Math. J. **5**, 552–566 (1939). [MF 174]

The concomitants of degrees 1, 2 and 3, together with their syzygies, all belonging to a ternary trilinear form F, in three independent sets of variables, are discussed. Next F is treated geometrically as a relation between three points x, y and z on three different planes. For a fixed x, points y answer to straight lines in the z plane. If the z lines are concurrent whenever the y points are collinear x must lie on a certain cubic curve $X(x)$. Thus we obtain three such cubics, one in each plane.

A study of the general case when each cubic is of genus unity leads to a canonical form of all four F, X, Y, Z. The relation between the two invariants of a ternary cubic and those of F is examined. Two such forms F_1 and F_2 are equivalent if, and only if, they have the same absolute invariants.

Certain quadratic transformations are considered between the cubics X, Y and Z, particularly when the transformations are periodic. This leads to an interesting connection between the sextactic points of a cubic and the vertices of the inscribed and escribed Hart triangles.

Light is also thrown on several earlier results, including those of Rosanes, Igel, and Maennchen.

H. W. Turnbull (St. Andrews, Scotland).

✶Turnbull, H. W. Theory of Equations. Oliver & Boyd, Edinburgh, 1939. xii+152 pp. 4/6.

The book gives a short and elementary account of algebraic equations, both from the theoretical and the practical side, together with the algebra of polynomials and rational fractions. It includes classes of number, partitions, identities, the G. C. M. process, partial fractions and recurring series, but it omits continued fractions and indeterminate equations. Cubic and biquadratic equations are discussed, together with more general types, elimination, and symmetric functions, but the theory of invariants and of groups is left untouched.

Bohlin, K. Suppléments à la théorie de l'équation algébrique du cinquième degré. Formules récentes pour les racines. Ark. Mat., Astr. Fys. **26**, no. 18, 22 pp. (1939). [MF 305]

Continuation of a long series of articles by the same author on the solution of equations of the fifth degree. They are inter-connected, and an independent brief résumé of the present paper is hard to give.

As standard form $x^5+5u^{-2}x=-u^{-5}-27$ is chosen. Corresponding standard forms for degrees 3 and 4 are established and the three equations compared. Much use is made of "racinal" expansions of the roots x of the quintic, that is, power-series around $u^{-1}=\infty$. The method is applied to certain numerical equations, with determination of errors.

A. J. Kempner (Boulder, Colo.).

Rosenbaum, Benjamin. On the irreducibility of certain classes of polynomials. Amer. J. Math. **61**, 923–933 (1939). [MF 287]

In two papers [S.-B. Preuss. Akad. Wiss. **1929**, 125, 370] I. Schur proved the irreducibility of the following polynomials in the field of rational numbers:

$$f_1(x)=1+g_1\frac{x}{1!}+g_2\frac{x^2}{2!}+\cdots+g_{n-1}\frac{x^{n-1}}{(n-1)!}\pm\frac{x^n}{n!},$$

$$f_2(x)=1+g_1\frac{x^2}{1}+g_2\frac{x^4}{1\cdot 3}+\cdots +g_{n-1}\frac{x^{2n-2}}{1\cdot3\cdot5\cdots(2n-3)}\pm\frac{x^{2n}}{1\cdot3\cdot5\cdots(2n-1)},$$

where the g_i are arbitrary rational integers. In generalization of Schur's first result, the author proves the irreducibility in the field of rational numbers of the polynomial

$$f(x)=\frac{g_0}{d_0}+g_1\frac{x^r}{d_1(s-t)!}+g_2\frac{x^{2r}}{d_2(2s-t)!}+\cdots+g_n\frac{x^{nr}}{d_n(ns-t)!},$$

where the following conditions have to be satisfied: The d_j, g_j, n, r, s and t are rational integers, $n>0$, $r>0$, $s>0$, $ns-2\geqq t\geqq 0$, and $n\geqq 2$ when $r\geqq 2$. The prime factors of g_0 and g_n are either greater than $ns-t$ or smaller than the largest prime which divides $ns-t$. The number d_0 divides $(p-1)!$, where p is the largest prime not greater than $ns-t$, and d_j is the least positive integer for which $d_j(js-t)!$ is divisible by d_0. (If $js-t<1$, $(js-t)!$ is to be set equal to 1.) In a similar manner, Schur's second result is generalized. As in Schur's paper, the proof is based on the theorem of Sylvester that any set of k consecutive integers not less than $k+1$ contains at least one integer divisible by a prime not less than $k+1$. Further, certain cases are studied in which Koenigsberger's criterion gives better results than the one mentioned above. *R. Brauer* (Toronto, Ont.).

Kneser, Hellmuth. Eine merkwürdige Mittelbildung bei algebraischen Gleichungen mit lauter positiven Wurzeln. Math. Z. **45**, 590–594 (1939). [MF 411]

Galerkin, and recently Grammel, have developed procedures for the numerical calculation of eigenvalues of positive definite operators by characterizing approximations for the first n eigenvalues as roots of algebraic equations. These equations are formed as suitable linear combinations of all the equations whose roots are n of the eigenvalues; in Grammel's procedure the linear combination differs from that in Galerkin's method. Kneser proves that Galerkin's values are upper bounds of Grammel's and that the latter are upper bounds of the first n eigenvalues, a theorem which was stated without complete proof by Grammel in a paper on his method, soon to appear in the Ingenieur-Archiv. Kneser's proof is based on the possibility of characterizing the approximate values as eigenvalues of quadratic forms and on the possibility of comparing such eigenvalues for different problems by means of their maximum-minimum-properties. *R. Courant* (New York, N. Y.).

Behrend, Felix. Über Systeme reeller algebraischer Gleichungen. Compositio Math. **7**, 1–19 (1939). [MF 367]

Man betrachte ein System von Formen

$$f_i(x_1,\cdots,x_r;y_1,\cdots,y_s),\qquad i=1,\cdots,n,$$

mit reellen Koeffizienten und Homogen von ungeraden Graden in jeder der beiden Variablenreihen x und y; das System heisse "definit," wenn das Gleichungssystem $f_1=0,\cdots,f_n=0$ im Reellen keine anderen Nullstellen hat als diejenigen mit $x_1=\cdots=x_r=0$ oder mit $y_1=\cdots=y_s=0$; ein Beispiel, mit $r=s=n=2$, ist das System $x_1y_1-x_2y_2$, $x_1y_2+x_2y_1$. Die Frage ist: welches ist, bei gegebenen r und s, die kleinste Zahl n, für die es ein definites System von n Formen der genannten Art gibt? Diese Mindestzahl heisse $n^*(r,s)$. Man sieht leicht, dass $\max(r,s)\leqq n^*(r,s)\leqq r+s-1$ ist. Der nachstehende Satz liefert für viele Fälle wesentlich bessere Abschätzungen nach unten für n^*: Satz: Dafür, dass es ein definites System von n Formen der genannten Art gibt, ist notwendig, dass alle Binomialkoeffizienten $C_{n,k}$ mit $n-r<k<s$ gerade sind. Eine der zahlreichen Konsequenzen des Satzes ist die, dass höchstens dann $n^*(r,r)=r$ sein kann, wenn r eine Potenz von 2 ist. Der Verfasser teilt mit, dass E. Stiefel und der Referent auf topologischen Wegen zu dem obigen Satz gelangt sind [veröffentlicht ist bisher nur ein kleiner Spezialfall von E. Stiefel, Comment. Math. Helv. **8**, 46–47 (1936)] und dass sie ihn auf die, von ihnen und anderen ohne Erfolg angegriffene, Aufgabe aufmerksam gemacht haben, für diesen algebraischen Satz auch einen "algebraischen" Beweis zu finden. Diese Aufgabe ist in der vorliegenden Arbeit in präzisem Sinne gelöst worden: der Beweis des Verfassers behält seine Gültigkeit, wenn man für die Koeffizienten und Variablen der Formen statt der reellen Zahlen einen beliebigen reell-abgeschlossenen Körper im Sinne von Artin-

Schreier zugrundelegt. Die Methode ist die der algebraischen Geometrie; sie benutzt die neuen Arbeiten von van der Waerden; für den Fall bilinearer Formen wird noch ein zweiter, ebenfalls algebraisch-geometrischer, Beweis angegeben, der an die Apolaritäts-Theorie von Reye anknüpft. Ferner gibt der Verfasser, mittels arithmetisch-algebraischer Überlegungen, auch neue Abschätzungen von n^* nach oben an; damit gelingt unter anderem die genaue Bestimmung von $n^*(r, r)$ für $r \leqq 8$. *H. Hopf* (Zürich).

Dieudonné, J. L'aspect qualitatif de la théorie analytique des polynomes. Ann. of Math. **40**, 748–754 (1939). [MF 319]

Let F be a certain set of polynomials $P(x)=\sum a_\nu x^\nu$ of given degree N (if $a_N=a_{N-1}=\cdots=a_{N-k+1}=0$, we say that $x=\infty$ is a zero of multiplicity k), such that with $P(x)$ also $cP(x)$, c constant, belongs to F. Let the corresponding set $(a_0, a_1, \cdots, a_N)$ of the complex projective space of N dimensions be closed. The author's problem is to determine the maximum ρ of all integers r such that there exists a set E_r of the plane (not identical with the whole plane) in which every polynomial of F has at least r zeros. Let $\lambda(x_0)$ be the maximal multiplicity with which the polynomials of F have x_0 as a zero, and let μ be the minimum of $\lambda(x_0)$ if x_0 runs over the whole plane. The main theorem is: $\rho=N-\mu$. The author applies this in various cases, some of them well known. For instance, the set of polynomials $a_1P_1(x)+\cdots+a_lP_l(x)$ is considered, where $P_\nu(x)$ are given polynomials of degree $n \geqq l$ whose roots are located in preassigned circular domains. *G. Szegö* (Stanford University, Calif.).

Turan, P. Über die Ableitung von Polynomen. Compositio Math. **7**, 89–95 (1939). [MF 369]

Let $\max|f(z)|=M$, $\max|f'(z)|=M'$, where z belongs to a certain domain B. The classical theorems of A. Markoff type furnish uniform upper bounds for M/M' when $f(z)$ runs over the set of all polynomials of a fixed degree n. This (precise) upper bound is n if B is the unit circle $|z| \leqq 1$, and n^2 if B is the segment $[-1, +1]$. The author is trying to find uniform lower bounds for M/M' provided $f(z)$ runs over the set of all polynomials of the given degree n whose roots are situated in B. He obtains in case $|z| \leqq 1$ the lower bound $n/2$, and in case $[-1, +1]$ the lower bound $\frac{1}{6}n^{1/2}$. This is precise in the first case, and it is precise in the asymptotic sense in the second one.
G. Szegö (Stanford University, Calif.).

Abstract Algebra

Klein-Barmen, Fritz. Zur Theorie der Gefüge und Vereine. Math. Z. **45**, 595–606 (1939). [MF 412]

The author discusses nine generalizations of the concept of lattice, six of which are self-dual. He shows that these are effectively different, by examples. He then partially orders them with respect to their generality.
G. Birkhoff (Cambridge, Mass.).

Dilworth, R. P. Non-commutative residuated lattices. Trans. Amer. Math. Soc. **46**, 426–444 (1939). [MF 473]

The author gives a set of postulates for the algebra of ideals under union, intersection, and (not necessarily commutative) multiplication, and a second set involving union, intersection, and "residuation." A lattice L with the properties postulated is called "residuated." The author proves first that connected chains of indecomposable elements under the unit element generate an "arithmetical" sublattice, one which is the direct product of chains. Further, if L is modular (as in rings for right-, left-, or two-sided ideals), and if every product ab contains some intersection $a \cap b^\nu$, then every "irreducible" element is "primary." Various other results are proved, most of them analogues of known results in the ideal theory of rings and hypercomplex algebras. For instance, a complemented modular residuated lattice with radical 0 is a Boolean algebra.
G. Birkhoff (Cambridge, Mass.).

McCoy, Neal H. A theorem on matrices over a commutative ring. Bull. Amer. Math. Soc. **45**, 740–744 (1939). [MF 336]

Let S be the ring generated by the nth order matrices $I, A_1, \cdots, A_m$ with elements in a commutative ring R with unit element. Let $\mathfrak{C}$ denote the two-sided ideal in S generated by the matrices $A_iA_j-A_jA_i$. Let $\mathfrak{m}$ be the ideal of $R'=R[x_1, \cdots, x_m]$ consisting of all elements of R' which correspond to 0 under the homomorphism $R'\sim S/\mathfrak{C}$. If $\mathfrak{p}_\alpha$ is a minimal prime ideal divisor of $\mathfrak{m}$, for every polynomial $f(A_1, \cdots, A_m)$ we have a prime ideal $\mathfrak{p}_\alpha'$ in $R[\lambda]$ consisting of the polynomials $t(\lambda)$ such that $t[f(x_1, \cdots, x_m)]\equiv 0$ $(\mathfrak{p}_\alpha)$. Let $\mathfrak{k}$ be the intersection of all $\mathfrak{p}_\alpha'$, and $\mathfrak{n}$ the minimum ideal of $f(A_1, \cdots, A_m)$. It is proved that, for every polynomial $f(A_1, \cdots, A_m)$, the radical of $\mathfrak{n}$ is $\mathfrak{k}$ if and only if every element of $\mathfrak{C}$ is nilpotent. If R is an algebraically closed field [N. H. McCoy, Bull. Amer. Math. Soc. **42**, 592–600 (1936), and **45**, 280–284 (1939)], this becomes the condition that the characteristic roots of every scalar polynomial $f(A_1, \cdots, A_m)$ be of the form $f(\lambda_1, \cdots, \lambda_m)$, where λ_i is a characteristic root of A_i. *C. C. MacDuffee.*

Krull, Wolfgang. Beiträge zur Arithmetik kommutativer Integritätsbereiche. VII. Inseparable Grundkörpererweiterung. Bemerkungen zur Körpertheorie. Math. Z. **45**, 319–334 (1939). [MF 44]

Let $\mathfrak{R}$ be an integrally closed ring whose quotient field $\mathfrak{K}$ has prime characteristic. The relatively algebraically closed subfield of $\mathfrak{R}$ shall be called K. The author investigates the decomposition of prime ideals $\mathfrak{p} \subset \mathfrak{R}$ in the extended ring $\mathfrak{S}=\mathfrak{R}\Lambda$, where Λ denotes an algebraic extension of K. If Λ is a radical field over K, that is, $\alpha^{p^f}-a=0$, $a \varepsilon$K, for every $\alpha \varepsilon \Lambda$, then $\mathfrak{p}$ is divisible by a single prime ideal $\mathfrak{q}$ of $\mathfrak{S}$. In order to investigate arbitrary extensions it is necessary to introduce the following definition. The field $\mathfrak{K}$ is called a strongly transcendental extension of K if $\mathfrak{K}\Lambda$ contains no proper algebraic elements over Λ, where Λ denotes an arbitrary algebraic extension of K. Under this hypothesis for $\mathfrak{K}$, K the following statements are true: (α) $\mathfrak{a}\mathfrak{S} \cap \mathfrak{R}=\mathfrak{a}$ for every ideal $\mathfrak{a}$ of $\mathfrak{R}$; (β) if $\mathfrak{a}=\mathfrak{b}_1 \cap \cdots \cap \mathfrak{b}_s$ in $\mathfrak{R}$, then $\mathfrak{a}\mathfrak{S}=(\mathfrak{b}_1\mathfrak{S}) \cap \cdots \cap (\mathfrak{b}_s\mathfrak{S})$ in $\mathfrak{S}$; (γ) if $\mathfrak{r}$ is a primary ideal belonging to $\mathfrak{p}$ in $\mathfrak{R}$, then $\mathfrak{r}\mathfrak{S}=\mathfrak{y}_1 \cap \mathfrak{y}_2 \cap \cdots$, where the primary ideals $\mathfrak{y}_i$ belong to the prime ideals $\mathfrak{q}_i \supset \mathfrak{p}\mathfrak{S}$. The proof is obtained by reduction to simple extensions $\mathfrak{K}(\alpha)$. In the latter cases the existence of the module representation $\mathfrak{S}=1\cdot\mathfrak{R}+\alpha\cdot\mathfrak{R}+\cdots+\alpha^{m-1}\cdot\mathfrak{R}$ immediately yields the conclusion. An analysis of the proof yields that statements (α), (β), (γ) are true for arbitrary $\mathfrak{R}$, $\mathfrak{K}$ if only simple extensions $\mathfrak{K}(\alpha)$ are admitted for the construction of $\mathfrak{S}$. It is shown by a counterexample that (α), (β) are in general not true if $\mathfrak{K}$ is not strongly transcendental over K. Let $\mathfrak{p}$ be a maximal prime ideal of $\mathfrak{R}$, that is, $\mathfrak{R}|\mathfrak{p}$ is a field $\bar{\Lambda} \supseteqq \Lambda$. (In order to determine the number of the $\mathfrak{q} \supset \mathfrak{S}\mathfrak{p}$ it suffices to consider maxi-

mal prime ideals, since every $\mathfrak{p}$ is maximal in its quotient ring.) Let $\Lambda^* \supset K$ be the maximal separable subfield of Λ; $\mathfrak{S}^* = \mathfrak{R}\Lambda^*$. Then the primes $\mathfrak{q} \supset \mathfrak{p}\mathfrak{S}$ are in (1-1)-correspondence with the primes $\mathfrak{q}^* \supset \mathfrak{p}\mathfrak{S}^*$. This theorem is a trivial corollary of the theorem on radical extensions. Thus the enumeration of the $\mathfrak{q} \supset \mathfrak{p}\mathfrak{S}$ is reduced to the case of separable extensions $\Lambda \supset K$. If Λ is a normal extension of K, then the number of $\mathfrak{q} \supset \mathfrak{p}\mathfrak{S}$ is equal to $[\Lambda \cap \bar{\Lambda} : K]$. This theorem again is proved by reduction to finite extensions $K(\xi)$. An essential tool in the latter case is found in the following lemma: Let $p(x) = p_1(x)^{r_1} \cdots p_s(x)^{r_s}$ be the decomposition in $\bar{\Lambda}$ of the irreducible polynomial defining $K(\xi)$. Then $\mathfrak{q}_i = (\mathfrak{p} + (p_i(x))) \cdot \mathfrak{S}$ are all the prime divisors of $\mathfrak{p}\mathfrak{S}$; the residue class field $\mathfrak{S} | \mathfrak{q}_i$ is equal to $\Lambda(\eta_i)$, where $p_i(\eta_i) = 0$. The isolated primary component of $\mathfrak{p}\mathfrak{S}$ which belongs to $\mathfrak{q}_i$ is distinct from $\mathfrak{q}_i$ if and only if $r_i > 0$. Incidentally, the preceding theorem and lemma are extremely useful in the arithmetic theory of base conditions in algebraic geometry. Finally the author investigates conditions in order that $\mathfrak{R}$ be strongly transcendental over K. *O. F. G. Schilling.*

Schilling, O. F. G. Units in p-adic algebras. Amer. J. Math. 61, 883–896 (1939). [MF 284]

The author considers the unit groups of maximal orders in simple p-adic algebras. The fact that every such algebra is complete with respect to a suitably defined pseudo-valuation simplifies the theory and gives the main result that every unit of a given maximal order can be expressed as an infinite product of a finite number of basis units and these products are convergent in the pseudo-valuation. The properties of the pseudo-valuation yield the result that the unit groups are totally disconnected locally compact groups, and every unit which lies in a sufficiently small neighborhood of the group unit can be expressed by an exponential function. *O. Ore* (New Haven, Conn.).

Moriya, Mikao. Über die Divisorenklassen nullten Grades in einem abstrakten elliptischen Funktionenkörper. J. Reine Angew. Math. 181, 61–67 (1939). [MF 588]

The author gives a new proof for a special case of the following theorem: Let K be an algebraic function field of one variable of genus 1 over an algebraically closed field of constants k with characteristic p. The number h_n of classes C of K with $C^n = 1$ is given as $h_n = n^2$ if $p \neq n$; $h_n = n$ if $n = p$ and $A = 0$; $h_n = 1$ if $n = p$ and $A \neq 0$ ($p \neq 0$). Here A denotes the invariant of Hasse [J. Reine Angew. Math. 174, 55–62 (1936)]. In this paper it is assumed that k is absolutely algebraic over a prime field of characteristic $p \neq 0$. The essential idea of the author's treatment is to reduce K to suitable subfields $\bar{K} \subset K$ with finite coefficient fields. Thus, the methods of class field theory (invariant classes and principal genus theorem) can be applied. The author derives relations between the l-class number of $\bar{K}$ and its cyclic ramified extensions Z of degree l. In order to prove the existence of such fields $Z/\bar{K}$, it is necessary to employ the existence of a class $C \neq 1$ in K with $C^l = 1$. Only at this stage the theory of differential determinants is required. As a matter of fact, it suffices to know that the latter is not a constant. The theorem is gotten by combining the information on suitable approximating fields $\bar{K}$ (finite class groups) with the existence theorem. *O. F. G. Schilling.*

THEORY OF NUMBERS

✶Wright, Harry N. First Course in Theory of Numbers. John Wiley and Sons, Inc., New York, 1939. vii+108 pp. $2.00.

Contents: I. Divisibility. II. Simple continued fractions. III. Congruences. IV. Quadratic residues. V. Diophantine equations. Table of primes ($\leqq 2741$).

✶Uspensky, J. V. and Heaslet, M. A. Elementary Number Theory. McGraw-Hill Book Company, Inc., New York, 1939. x+484 pp. $4.00.

This text is intended to introduce to the reader the more interesting and striking of the many theorems, proved or conjectured, that make up what is known as "elementary" number theory. In order to make more realistic some of the general principles, the authors have introduced topics from recreational mathematics as appendices to some of the chapters. Topics conspicuous by their absence are the continued fraction and the Gaussian theory of quadratic forms.

The first four chapters of the book deal with the elementary and divisibility properties of numbers, Euclid's algorithm, and properties of primes. The discussion of Euclid's algorithm and its application is unusually complete, giving Lamé's theorem on the number of operations required in finding the greatest common divisor of two numbers, as well as an account of the least remainder algorithm. The chapter on primes is entirely elementary (the prime number theorem and the magnitude of its remainder are stated and illustrated with a table) and contains also discussions of the totient function and sum of divisors together with some applications of the greatest integer function. Chapter 5 is devoted to what Sylvester called the "principle of cross-classification" and its application. The properties of the Möbius inversion function and Meissel's exact formula for the number of primes not exceeding x are among the applications considered.

The theory of congruences is introduced in Chapter 6, which has an appendix consisting of a short discussion on the use of this theory in the construction of magic squares. Chapter 7 deals with congruences in one unknown, the emphasis being laid on Lagrange's theorem concerning the number of roots of a congruence with a prime modulus, and its applications. There is also a derivation of Voronoi's interesting formula

$$x = 3 - 2a + 6 \sum_{h=1}^{a-1} \left[\frac{mh}{a}\right]^2$$

for the solution of the linear congruence $ax \equiv 1 \pmod{m}$. An appendix gives a thorough discussion of Gauss' rule for the date of Easter. Chapter 8, entitled Residues of Powers, contains the usual material on the binomial congruence and has an appendix on card shuffling in which the riffle and Monge's shuffle are analysed. The arithmetic properties of Bernoulli numbers is the subject of Chapter 9. Here are proved the theorems of von Staudt, Adams, and Kummer. The last two and also numerous exercises are made to follow from Voronoi's congruence

$$(a^{2k}-1)P_k \equiv (-1)^{k-1} 2ka^{2k-1} Q_k \sum_{S=1}^{N-1} S^{2k-1}[Sa/N] \pmod{N},$$

where P_k/Q_k is the kth Bernoulli number and a is any integer prime to the arbitrary number N. Chapter 10 gives a very full discussion of quadratic residues from Euler's

criterion to Gauss' method of exclusion. Gauss' fifth proof of the reciprocity law is given. The indetermined equation $t^2 \pm Du^2 = m$ forms the chief topic of Chapter 11. The case of $m=1$ is treated without continued fractions. There is a discussion of Chebyshev's results on the problem of representing m by $t^2 - Du^2$, together with tables for applying the method of exclusion to this problem. Kummer's proof of the reciprocity law depending on solutions of the Pell equation is also given. The chapter concludes with Dickson's proof of the four square theorem of Fermat. Chapter 12, entitled Some Diophantine Problems, treats several such equations, due mostly to Fermat, including $x^2+ay^2=z^n$, $x^2+c=y^3$, $x^4-y^4=z^2$, $x^4-2y^2=\pm 1$, $x^3+y^3=z^3$, and the system $x^2+5y^2=z^2$, $x^2-5y^2=t^2$.

The final chapter deals with several theorems from what is sometimes called the arithmetic theory of elliptic functions. The treatment here is entirely elementary however, based as it is on Liouville's fundamental identities for arbitrary parity functions. Proofs are given of Euler's recurrence formula for the sum of the divisors of a number and of other similar formulas for functions depending on divisors. In conclusion there is a complete elementary treatment of the problems of representing a number as the sum of 3 squares, 4 squares, and 3 triangular numbers. The last 6 pages of the book contain a list of primes to 5000 and a table of indices and corresponding numbers for prime moduli less than 100, as arranged by Wertheim in his Anfangsgründe der Zahlentheorie [Brunswick, 1902]. The treatment throughout the book is free from obscurities and pedantry and yet is never lacking in rigor or interest.

D. H. Lehmer (Bethlehem, Pa.).

Travers, J. Rules for bordered magic squares. Math. Gaz. 23, 349–351 (1939). [MF 354]

Fitting, F. Pandiagonale Quadrate von $(4m)^2$ Feldern. Nieuw Arch. Wiskde 20, 55–58 (1939). [MF 568]

Feldheim, Ervin. Rectification à la note "un problème de la théorie élémentaire des nombres." Bull. Soc. Math. France 67, 100–101 (1939). [MF 447]

Concerning the paper quoted [Bull. Soc. Math. France 66, 1–7 (1938)].

Erdös, P. Note on the product of consecutive integers (II). J. London Math. Soc. 14, 245–249 (1939). [MF 428]

The author proves that for every $l>2$ there exists a $k_0=k_0(l)$, such that, when $k \geqq k_0$, the product of k consecutive integers cannot be a perfect lth power. He had previously shown that such a product is never a perfect square [J. London Math. Soc. 14, 194–198 (1939); Math. Reviews 1, 4]. The method used here is very similar to that used in the earlier paper; in addition the well-known theorem of Thue is employed at one point. The author also shows in a very simple way that the binomial coefficient $C_{n,k}$ $(n \geqq 2k)$ is not a perfect lth power if $k \geqq 2^l$ $(l>1)$ or if $l=3$ $(k>1)$. It would be possible to prove this for any $l \geqq 3$ $(k>1)$ if one could show that the equations

$$x^l \pm 1 = 2y^l, \quad x^l \pm 1 = 2^{l-1} y^l$$

have no solution in positive integers (except $x=y=1$ for the first equation). *H. W. Brinkmann* (Swarthmore, Pa.).

Gloden, A. Sur le système diophantien $x+y+z=u+v+w$, $xyz=uvw$. Bol. Mat. 12, 205–209 (1939). [MF 458]

Mahler, K. A proof of Hurwitz's theorem. Mathematica, Zutphen. B. 8, 57–61 (1939). [MF 87]

A simple proof of the following theorem: Let a, b, c, d be four real numbers of determinant $ad-bc=1$, ϵ a positive number. Then there are two integers u and v, such that

$$|(au+bv)(cu+dv)| \leqq 1/\sqrt{5}, \quad |au+bv| < \epsilon; \; u^2+v^2>0.$$

If

$$a=\frac{1+\sqrt{5}}{2\sqrt[4]{5}}, \quad b=\frac{1}{\sqrt[4]{5}}, \quad c=\frac{1-\sqrt{5}}{2\sqrt[4]{5}}, \quad d=\frac{1}{\sqrt[4]{5}},$$

then

$$|(au+bv)(cu+dv)| \geqq 1/\sqrt{5}$$

for all integers u and v that do not vanish simultaneously. *J. F. Koksma* (Amsterdam).

van der Corput, J. G. Sur un certain système de congruences. I. Nederl. Akad. Wetensch., Proc. 42, 538–546 (1939). [MF 310]

Let N be the number of incongruent solutions $y_1, \cdots, y_n$ of

$$\text{(I)} \qquad \sum_{\nu=1}^{n} b_{\mu\nu}\psi_\nu(y_\nu) \equiv g_\mu (\text{mod } v_\mu), \quad \mu=1,2,\cdots,m,$$

where all $v_\mu = p^\alpha$, and α, $b_{\mu\nu}$, g_μ are integers, $\psi_\nu(y)$ polynomials with integral coefficients such that the derivatives $\psi_\nu'(y)$ have no zeros of higher multiplicities than s. The author intends to prove the following theorems: (1) If $n \geqq (s+2)m$, and if for every positive integer $\mu \leqq m$ the matrix $B=(b_{\mu\nu})_{\mu=1,2,\cdots,m;\,\nu=1,2,\cdots,n}$ retains at least the rank $m+1-\mu$ if $(s+2)\mu-1$ arbitrary columns are suppressed, then $N \leqq cp^{(n-m)\alpha}$, where c depends only on B and the $\psi_\nu(y)$. (2) The same conditions as under (1) are satisfied, but $v_1, \cdots, v_m$ are now arbitrary positive integers of product not greater than an integer X, and N is the number of solutions of (I) in positive integers $y_\nu \leqq X$. Then

$$N \leqq 2^n \tau(v_1 v_2 \cdots v_m)^\eta X^n (v_1 v_2 \cdots v_m)^{-1},$$

where $\tau(w)$ is the number of divisors of w, and η a constant depending only on B and the $\psi_\nu(y)$. (3) The less stringent conditions are made that $n \geqq m$ and that the rank of B is m. Then if N has the same meaning as under (2)

$$N \leqq 2^n X^n \prod_{\mu=1}^{m} \prod_{p|v_\mu} \omega p^{-1-(1/m)[\alpha/(s+2)]},$$

where p^α is the highest power of p which divides v_μ, and ω is a suitable number which depends alone on B and the $\psi_\nu(y)$. In this first note, the author begins with four lemmas which are necessary for the proof of (1). The fourth lemma gives upper bounds for the number of solutions of $\psi(y) \equiv g \pmod{p^\alpha}$. *K. Mahler* (Manchester).

O'Connor, R. E., S.J. Quadratic and linear congruence. Bull. Amer. Math. Soc. 45, 792–798 (1939). [MF 345]

This paper considers the number $N(p^m)$ of simultaneous solutions of a quadratic congruence

$$f(x) = \sum_1^n a_{ij} x_i x_j \equiv r \pmod{p^m}$$

and a linear congruence

$$g(x) = \sum_1^n c_i x_i \equiv s \pmod{p^m},$$

where f and g are integral forms, $n \geqq 2$, r and s are integers, and p is an odd prime. The author employs Minkowski's canonical form $F(x) = \sum_1^n a_i x_i^2$ of $f(x)$ [Minkowski: Ge-

sammelte Abhandlungen, vol. 1, p. 14] and also a theorem of Jordan [C. R. Acad. Sci. Paris **62**, 687 (1866)] concerning the number of solutions of an integral quadratic congruence $\sum d_i x_i^2 \equiv h \pmod{p}$, to determine $N(p^m)$, for $m=1$, in terms of quadratic character symbols associated with the above forms. The author distinguishes between ordinary and singular solutions of the above system (a singular solution is a solution ξ such that, for some integer λ, p divides each of $\sum_j (a_{ij}\xi_j) - \lambda c_i$ for $i=1, \cdots, n$), determines the number $M(p^m)$ of the ordinary solutions for $m \geqq 1$, and indicates instances where these are the only solutions. An application to certain integral vectors is suggested in the last section. *A. E. Ross* (St. Louis, Mo.).

Brauer, Alfred. On addition chains. Bull. Amer. Math. Soc. **45**, 736–739 (1939). [MF 335]

Let $l(n)$ denote the least l for which a chain of increasing integers $a_0=1$, $a_1=2$, a_2, $\cdots$, $a_l=n$ exists, such that each a_ρ is a sum $a_\sigma + a_\tau$ of preceding terms of the chain. A. Scholz [Jber. Deutsch. Math. Verein. **47**, 41 (1937)] stated that: (1) $m+1 \leqq l(n) \leqq 2m$ if $2^m+1 \leqq n \leqq 2^{m+1}$ $(m \geqq 1)$; (2) $l(ab) \leqq l(a)+l(b)$; (3) $l(2^{m+1}-1) \leqq m + l(m+1)$. Brauer proves (1) and (2), and that for large n,

$$l(n) < (\log n/\log 2)\{1+(\log\log n)^{-1}+2(\log 2)(\log n)^{\log 2-1}\},$$

whence $l(n) \sim (\log n)/(\log 2)$. As regards (3) he proves $l(2^{m+1}-1) \leqq m + l^*(m+1)$, where $l^*(n)$ is the minimum length of only those chains in which every a_ρ is a sum $a_{\rho-1}+a_\tau$. Also, if r is a positive and s a non-negative integer, $l(n) \leqq (r+1)s+2^r-2$ if $2^{rs} \leqq n < 2^{r(s+1)}$. *G. Pall.*

Atkinson, F. V. A summation formula for $p(n)$, the partition function. J. London Math. Soc. **14**, 175–184 (1939). [MF 240]

With $p(n)$ denoting the number of unrestricted partitions of n, the author first proves the summation formula:

$$(1)\qquad \sum_{n-1/24<x} p(n)\frac{\{2\pi(x-n+1/24)\}^k}{\Gamma(1+k)} = \sum_{n=0}^{\infty} p(n)\left(\frac{x}{n-1/24}\right)^{\frac12 k-\frac14} J_{k-\frac12}\{4\pi\surd(x(n-1/24))\},$$

valid if $k > -\frac12$, $x > 1/24$, and $x+1/24$ not an integer. The series on the right-hand side is Abel summable, but not summable by arithmetic means of any order.

The formula (1) appears in the paper as a consequence of the functional equation

$$(2)\qquad \sum_{n=0}^{\infty} p(n)e^{-2\pi(n-1/24)z} = \sqrt{z}\sum_{n=0}^{\infty} p(n)e^{-2\pi(n-1/24)/z},$$

which itself is a special instance of the transformation theory of the modular form $\eta(\tau)^{-1}$. In the same manner as (2) belongs to the point $z=1$, there exist infinitely many other similar functional equations (transformation formulae), each belonging to a root of unity. Each of those, as the author points out, will give rise to a summation formula analogous to (1).

The proof of (1) is a rather direct one and operates with a term by term integration over $1-i\infty$ to $1+i\infty$. The difficulties lie in the justification of certain interchanges of limits. For $k=0$ the formula (1) can be written

$$(3)\qquad \sum_{n+1/24<x} p(n) = \lim_{h\to\infty} s(h, x),$$

with

$$(3\text{a})\qquad s(h, x) = \sum_{n=0}^{\infty} p(n)e^{-2\pi(n-1/24)/h}\frac{1}{\pi(2x)^{1/2}}\cos 4\pi\surd(x(n-1/24)).$$

The author proves that, for x in the interval (a, b), $1/24 < a < b$, the series $s(h, x)$ remains bounded if $h \to \infty$. Multiplication of (3) by the integrable derivative $f'(x+1/24)$ and termwise integration lead to the result: If $f(x+1/24)$ has an integrable derivative in the closed interval (a, b), where $b > a > 1/24$, and neither $a+1/24$ nor $b+1/24$ is an integer, then

$$\sum_{a<n-1/24<b} p(n)f(n) = \frac{1}{\pi\sqrt{2}}\sum_{n=0}^{\infty} p(n)\int_a^b f(x+1/24)\frac{d}{dx}\{x^{-\frac12}\cos 4\pi\surd(x(n-1/24))\},$$

the series being Abel summable. (The integrability of $f'(x+1/24)$ is not included in the author's own statement of his theorem, but is explicitly mentioned in the course of the proof.) *H. Rademacher* (Swarthmore, Pa.).

Erdös, Paul and Wintner, Aurel. Additive arithmetical functions and statistical independence. Amer. J. Math. **61**, 713–721 (1939). [MF 29]

A real arithmetical function $f(n)$ (that is, a function defined for $n=1, 2, \cdots$) is called additive if $f(n_1n_2)=f(n_1)+f(n_2)$ whenever $(n_1, n_2)=1$. If there exists a monotone function $\sigma=\sigma(x)$, $-\infty < x < +\infty$, satisfying the boundary conditions $\sigma(-\infty)=0$, $\sigma(+\infty)=1$, such that for any continuity point of $\sigma(x)$ the set of n-values at which $f(n) < x$ has a relative density represented by $\sigma(x)$, then f is said to have an asymptotic distribution function σ. The problem of finding conditions for the existence of $\sigma(x)$ has been treated by Schoenberg and Davenport by statistical methods (moments and Fourier transforms) and by Erdös by elementary methods. The present paper contains a complete solution of this problem: (i) Placing $y^+=y$ or $y^+=1$ according as $|y|<1$ or $|y| \geqq 1$, a necessary and sufficient condition for the existence of $\sigma(x)$ is the convergence of the two series $\sum f^+(p)/p$ and $\sum f^+(p)^2/p$, where the summation is over all primes p. The sufficiency of this condition had already been proved by Erdös; its necessity also follows by elementary considerations. An additive arithmetical function always satisfies the condition $f(1)=0$; it is uniquely determined by its values on prime powers, and these values may be arbitrarily chosen. Denoting by $f^{(k)}(n)$ the additive function which coincides with $f(n)$ on the powers of the kth prime p_k and is zero on all other prime powers, one has $f(n)=\sum_{k=1}^{\infty} f^{(k)}(n)$. Evidently every function $f^{(k)}(n)$ possesses an asymptotic distribution function $\sigma^{(k)}=\sigma^{(k)}(x)$, namely the step function which has at $x=f(p_k^l)$ the jump $p_k^{-l}-p_k^{-(l+1)}$. It is also easily seen that, for every k, the additive function $f_k(n)=\sum_{j=1}^{k} f^{(j)}(n)$ possesses an asymptotic distribution function $\sigma_k=\sigma_k(x)$, which is the convolution $\sigma_k=\sigma^{(1)}*\cdots*\sigma^{(k)}$. Combining these facts with known results, one is led to the following theorems: (ii) A necessary and sufficient condition for the existence of $\sigma(x)$ is that $f_k(n)$ converges to $f(n)$ in relative measure, in which case $\sigma_k \to \sigma$. (iii) A necessary and sufficient condition for the existence of $\sigma(x)$ is that the infinite convolution $\sigma^{(1)}*\sigma^{(2)}*\cdots$ is convergent, in which case $\sigma^{(1)}*\sigma^{(2)}*\cdots=\sigma$. (iv) The function $\sigma(x)$ is, when it exists, either a step function, or singular, or absolutely continuous. (v) It is a step function if and only if the series $\sum_{f(p)\neq 0}(1/p)$ is convergent. For

results regarding the two other cases see the following paper. (vi) If $\sigma(x)$ exists, its spectrum is the closure of the set of values of $f(n)$. *B. Jessen* (Copenhagen).

Erdös, Paul. On the smoothness of the asymptotic distribution of additive arithmetical functions. Amer. J. Math. 61, 722–725 (1939). [MF 30]

Let $f(n)$ be an additive arithmetical function which possesses an asymptotic distribution function $\sigma=\sigma(x)$; as shown in the preceding paper, $\sigma(x)$ is either a step function, or singular, or absolutely continuous, and it is a step function if and only if the series $\sum_{f(p)\neq 0}(1/p)$ is convergent. The author now discusses certain extreme cases of functions $f(n)$ with continuous $\sigma(x)$ in which it is possible to distinguish between singular and absolutely continuous behavior of $\sigma(x)$. Let $f(p^l)=f(p)$ for all p and l. Sections 1 and 2 depend on estimates of the Fourier transform of $\sigma(x)$, which is given by

$$L(u)=\int_{-\infty}^{+\infty}e^{iux}d\sigma(x)=\Pi\left(1-\frac{1-\exp\,(if(p)u)}{p}\right).$$

If $f(p)=(-1)^{(p-1)/2}(\log\log p)^{-\alpha}$ $(\alpha>\frac{1}{2})$, a simple application of Merten's result

$$\prod_{p<t}(1-1/p)\sim e^{-\gamma}/\log t$$

leads to an estimate $L(u)=O(\exp-C|u|^{1/\alpha})$ for a positive constant C; hence $\sigma(x)$ may be continued to an entire function $\sigma(z)=\sigma(x+iy)$ if $\frac{1}{2}<\alpha<1$, to a function regular at least in the strip $|y|<C$ if $\alpha=1$, and has at least derivatives of arbitrarily high order if $\alpha>1$. If $f(p)=1/\log p$, one finds $L(u)=O(|u|^{-\frac{1}{2}-\epsilon})$ for some $\epsilon>0$; by Plancherel's theorem this implies the absolute continuity of $\sigma(x)$. If $f(p)=2^{-p}$, one finds $L(u)\neq o(|u|)$; hence $\sigma(x)$ is singular. In section 3 it is shown by elementary considerations that, if $f(p)=O(p^{-c})$ for some $c>0$, then $\sigma(x)$ is singular. This result is particularly interesting, since it contains the case $f(n)=\log n/\varphi(n)$, where $\varphi(n)$ is Euler's function. It may be mentioned that the assumption $f(p^l)=f(p)$ for all p and l is not necessary, since the character of $\sigma(x)$ depends only on the values of $f(n)$ on the primes. *B. Jessen.*

van der Corput, J. G. Une inégalité relative au nombre des diviseurs. Nederl. Akad. Wetensch., Proc. 42, 547–553 (1939). [MF 311]

The paper discusses principally sums of the type

$$S=\sum_{y=1}^{X}\tau^l(y)T(y),$$

where $\tau(y)$ is the number of divisors of y, l a natural number, and $T(y)\geqq 0$. The problem is to impose on $T(y)$ suitable and applicable conditions such that

$$S\leqq C\cdot(\log X)^{\omega}$$

for certain C and ω. Such a condition (that one assumed in Theorem 1) is that there exist three numbers A, γ, η such that for all positive integers $v\leqq X^\gamma$ we have

$$\sum_{\substack{y=1\\ y\equiv 0 \,(\mathrm{mod}\, v)}}^{X} T(y)\leqq A\cdot v^{-1}\tau^{\eta}(v).$$

A second theorem introduces a weaker but more complicated condition, which cannot conveniently be reproduced here. The proof of Theorem 2 (of which Theorem 1 is shown to be a special case) is based on a peculiar factorization of y.

By means of his Theorem 1 and a lemma due to Hua [J. London Math. Soc. 13, 54–61 (1938)] the author finally proves a third theorem: If $\psi(h)$ is a polynomial of integral coefficients and l a positive integer, then there exists a number Ω such that, for all $Z\geqq 3$,

$$\sum_{\substack{h=1\\ \psi(h)\neq 0}}^{Z}\tau^l(|\psi(h)|)\leqq Z(\log Z)^{\Omega}.$$

H. Rademacher (Swarthmore, Pa.).

Broderick, T. S. On obtaining an estimate of the frequency of the primes by means of the elementary properties of the integers. J. London Math. Soc. 14, 303–310 (1939). [MF 437]

The object of this paper is to prove a theorem, equivalent to that of Chebyshev on the distribution of primes, by methods which involve only the elementary properties of integers. Instead of the transcendental function $\log n$ the author uses the numerical function

$$\lambda(n)=\sum_{r\leqq n}r^{-1},$$

and proves that the number of primes not exceeding n satisfies, for all $n>1$, the inequalities

$$\frac{n}{20\lambda(n)}<\pi(n)<\frac{14n}{\lambda(n)}. \tag{1}$$

The method of proof follows the lines of the proof of Chebyshev's theorem as given in Hardy and Wright: The Theory of Numbers [Oxford, 1938]. Only slight complications attend the substitution of $\lambda(n)$ for $\log n$. For example we have

$$\lambda(mn)<\lambda(n)+\lambda(m)\leqq\lambda(mn)+1.$$

The general method is sufficient to establish the second inequality for $n>3$, but the first inequality only for $n\geqq 11^8$, so that it was necessary to verify the result directly for $n<11^8$. *D. H. Lehmer* (Bethlehem, Pa.).

Kienast, Alfred. Über die asymptotische Darstellung der summatorischen Funktion von Dirichletreihen mit positiven Koeffizienten. Math. Z. 45, 554–558 (1939). [MF 408]

A general asymptotic formula, obtained in a previous paper by an adaptation of Wiener's method [Math. Z. 44, 115–126 (1938)], is corrected by the addition of further terms in the case in which a certain parameter ρ is non-integral. The correction does not affect (for example) the application to the estimation of the error in the prime number theorem, which is now derived in the Titchmarsh form $e^{-\omega}\psi(e^{\omega})=1+O(e^{-p\omega^q})$, $p>0$, $q=5/9-\epsilon$. *A. E. Ingham* (Princeton, N. J.).

Davenport, H. On character sums in finite fields. Acta Math. 71, 99–121 (1939). [MF 212]

Let $\chi_1, \cdots, \chi_r$ be non-principal multiplicative characters in $GF(q)$, $q=p^n$; let $f_1(x), \cdots, f_r(x)$ be distinct irreducible normalized polynomials over $GF(q)$ of degree $k_1, \cdots, k_r$, respectively; let $k=k_1+\cdots+k_r$. Put

$$\sigma_\nu=\sum\chi_1\{(f_1, g)\}\cdots\chi_r\{(f_r, g)\},$$

summed over all normalized $g(x)$ of degree ν; (f, g) is the resultant. Then

$$L(f, \chi; s)=\sum_{\nu=0}^{\infty}\sigma_\nu q^{-\nu s}$$

is essentially Hasse's L-function [J. Reine Angew. Math.

172, 37–54 (1934)]. In this paper a functional equation for L is derived, and the zeros are studied. In certain cases ($k=2, 3$) it is proved that the zeros have real part equal to 1/2. In the general case it is proved that the real part σ satisfies $\theta_k \leqq \sigma \leqq 1-\theta_k$, where $\theta_3=1/4$, $\theta_k=3/(2(k+4))$, for $k\geqq 4$. Applications are made to the distribution of power-residues (mod p) and to the distribution of the irreducible polynomials (mod p) with respect to which a fixed polynomial is a primitive root. *L. Carlitz* (Durham, N. C.).

Davenport, H. On Waring's problem for fourth powers. Ann. of Math. 40, 731–747 (1939). [MF 318]

This paper gives a proof of the following theorem: Every sufficiently large integer is representable as a sum of 14 integral fourth powers unless it is congruent to 0 or 15 (mod 16). It follows that every sufficiently large integer is representable as a sum of 16 integral fourth powers. This is a best possible result, since an argument due to Kempner [Math. Ann. 72, 395–396 (1912)] shows that no integer of the form $16^k 31$ is representable as a sum of less than 16 integral fourth powers. The method of proof is quite similar to that given previously for the case of representation by cubes [Acta Math. 71, 123–143 (1939)] and the same new idea is used [Proc. Roy. Soc. London 170, 293–299 (1939)]. *R. D. James* (Saskatoon, Sask.).

Erdös, P. On the integers of the form x^k+y^k. J. London Math. Soc. 14, 250–254 (1939). [MF 429]

The object of this paper is to give an elementary proof of the following theorem: If k is odd and not less than 3, the number of integers not exceeding n of the form x^k+y^k, where x and y are relatively prime and positive, is greater than $C\cdot n^{2/k}$, C being positive and independent of n. This is a very special case of a theorem proved by the author and K. Mahler in a less elementary way [J. London Math. Soc. 13, 134–139 (1938)]. The proof as it stands is defective since one of the lemmas used (Lemma 3) is incorrect. It is possible, however, to fix up the proof by correcting this lemma and modifying the subsequent argument appropriately. *H. W. Brinkmann* (Swarthmore, Pa.).

THEORY OF GROUPS

★Weyl, Hermann. The Classical Groups. Their Invariants and Representations. Princeton University Press, Princeton, N. J., 1939. xii+302 pp. $4.00.

Les groupes classiques dont parle Monsieur Weyl sont les groupes linéaires général ($G\,L(n)$) et spécial ($S\,L(n)$), le groupe orthogonal $O(n)$ et le groupe symplectique ($Sp(n)$) (groupe d'un complexe linéaire). Le chapitre I, en forme d'introduction, pose les notions algébriques de base: corps, anneaux, polynômes, espaces linéaires. Le lecteur y trouvera également une explication détaillée du double aspect de la notion de groupe en mathématiques et même en physique mathématique: d'une part, groupe des automorphismes d'un ensemble préservant certaines relations (telles celles par exemple qu'introduit la géométrie euclidienne); d'autre part, groupe des transitions permettant de passer d'un système de référence à un autre. Cette étude consiste en somme en une mise au point du fameux programme d'Erlangen, complété de manière à fournir la théorie des quantités de divers types (vecteurs, tenseurs, . . .), dont chacune se transforme suivant sa loi propre. La notion de ces lois de transformation diverses relatives à un même groupe introduit la notion de représentation d'un groupe, celle-ci à son tour l'idée d'invariants de divers types.

Le chapitre II contient la détermination explicite par des moyens algébriques des invariants de type vectoriel (fonctions d'un certain nombre d'arguments vectoriels) des groupes $G\,L(n)$, $O(n)$. La méthode consiste à exprimer, au moyen d'identités dues à Capelli, tous les invariants au moyen de ceux qui ne dépendent que de $n-1$ arguments, puis à construire directement ces derniers. Dans chacun des cas envisagés, un "premier théorème principal" fournit les invariants tandis qu'un "second théorème principal" fournit les relations entre eux. L'étude des invariants du groupe $O(n)$ avec un corps de base quelconque est possible en vertu de l'existence de la paramétrisation de Cayley des substitutions orthogonales. Celle-ci à son tour permet de donner dès ce chapitre tout algébrique un aperçu de l'aide que fournissent les considérations infinitésimales de la théorie des groupes de Lie.

Le chapitre III contient essentiellement la théorie des algèbres associatives semi-simples, qui est traitée du point de vue de la théorie des représentations, suivant la méthode inventée par l'auteur. Le lecteur ne perd jamais de vue le lien avec la théorie des invariants et covariants: la théorie de l'algèbre des commutants (commutatoralgebra), dès qu'introduite, est mise en relation avec la notion de covariant; les anneaux de groupe sont introduits à partir d'une première esquisse des relations entre la réduction de l'espace des tenseurs et les représentations du groupe symétrique. De ce point de vue, on s'intéresse surtout à la réduction de l'algèbre des commutants d'une représentation d'un groupe fini: l'auteur procède à cette réduction directement, en montrant qu'elle se poursuit parallèlement à celle de l'anneau de groupe, sans passer par l'algèbre enveloppante de la reprèsentation elle-même.

Les moyens algébriques dont on dispose à ce stage permettent de tenir les promesses sur la foi desquelles les constructions antérieures ont été entreprises: les chapitres IV, V, VI contiennent l'exposé de la réduction complète des représentations tensorielles des groupes $G\,L(n)$, $O(n)$, $Sp(n)$. Dans le cas de $G\,L(n)$ on montre que les matrices de la représentation tensorielle de degré f sont celles de l'algèbre des commutants de la représentation correspondante du groupe symétrique; dans le cas de $O(n)$, on peut introduire une algèbre qui joue le même rôle que l'anneau du groupe symétrique; plutôt que d'en passer par l'étude de cette algèbre, on préfère opérer d'abord certaines réductions qui correspondent à l'opération de "rajeunissement" des tenseurs, puis s'aider des opérations induites par les éléments du groupe symétrique dans les sous-espaces qu'on obtient ainsi. Une méthode analogue conduit au but dans le cas de $Sp(n)$. Ici comme au chapitre II, la paramétrisation de Cayley permet de lever toute hypothèse restrictive sur le corps de base. On examine aussi le cas des représentations du groupe orthogonal propre. Par ailleurs ces chapitres contiennent une caractérisation complète des relations algébriques qui doivent exister entre les éléments d'une matrice pour qu'elle soit orthogonale ou symplectique.

À partir du chapitre VII, l'atmosphère générale du livre change quelque peu: l'analyse se substitue progressivement à l'algèbre, amenant dans ses bagages la topologie. On part du problème qui consiste à calculer les caractères des représentations irréductibles qu'on a déjà déterminées. Cette détermination repose sur le célèbre "artifice de l'unitarité"

de H. Weyl, qui permet de se ramener à des sous-groupes du groupe unitaire, donc compacts; dans ces groupes compacts, on peut développer des considérations intégrales grâce à l'existence d'un élément de volume invariant. De plus, on prouve de cette manière qu'il n'existe pas d'autres représentations irréductibles du groupe unitaire que celles déjà rencontrées, données par les tenseurs. L'auteur ne manque pas de recouper les résultats qu'il obtient pour $GL(n)$ en les déduisant à nouveau d'une manière purement algébrique, ce qui se trouve possible en raison de la connection avec le groupe symétrique; mais il n'a pas de peine à montrer combien ici la méthode analytique est plus attrayante. Le chapitre se termine sur la détermination des polynômes de Poincaré des groupes en question, dont les coefficients sont définis au moyen des représentations par les tenseurs anti-symétriques: ce n'est qu'au chapitre suivant qu'apparaîtra la signification topologique de ces polynômes.

Le chapitre VIII est consacré à la théorie générale des invariants (Rappelons que les invariants considérés au chapitre II étaient des invariants de type vectoriel, fonctions de un ou plusieurs vecteurs). La théorie classique ne considérait guère que les invariants des formes homogènes, c'est-à-dire les invariants dépendant de tenseurs symétriques. Le cadre de la recherche est élargi par l'auteur, qui se propose la détermination des invariants qui dépendent de "quantités" quelconques, définies par des représentations irréductibles. Ce problème peut être approché soit par des méthodes algébriques, soit par des méthodes analytiques. Côté algèbre, on trouve bien entendu la célèbre méthode symbolique, qui permet de ramener la recherche des invariants de degrès donnés à celle d'invariants de type vectoriel; ce n'est qu'aidée du dehors qu'elle peut quelquefois fournir une base finie pour tous les invariants. Pour générale que soit la méthode symbolique, elle n'est cependant pas la seule méthode algébrique qui puisse donner des résultats, comme il est montré par deux exemples importants. Pour obtenir une base finie du système des invariants de $GL(n)$, on utilise le théorème de Hilbert sur la base d'un idéal; il permet d'exprimer tout invariant comme combinaison de certains d'entre eux avec pour coefficients des polynômes qui ne sont pas nécessairement des invariants. Un artifice algébrique permet d'en déduire une expression à coefficients invariants. On peut déduire de là l'existence d'un nombre fini d'invariants de base pour certains autres groupes, tels notamment le groupe affine, par un procédé que l'auteur appelle "adjunction argument," qui consiste à introduire un ou plusieurs covariants qui expriment algébriquement la notion de l'"absolu" des géométries correspondantes. Côté analyse, on est tout naturellement amené à envisager l'aspect "groupes de Lie" des groupes en question, qui conduit à des équations différentielles pour les invariants. Par ailleurs, si on a affaire à un groupe compact, l'intégration sur le groupe fournit un procédé simple pour remplacer par des invariants les coefficients polynômes qui interviennent après emploi du théorème de Hilbert. D'autre part, l'artifice d'unitarité permet de déterminer toutes les représentations des groupes unimodulaires réel et complexe. Cette question conduit tout naturellement à une étude topologique des groupes classiques; dans le cas du groupe orthogonal, sa connexion particulière conduit à la notion de spineur.

Le chapitre IX et dernier reprend en la résumant et l'étendant l'étude des algèbres associatives qui faisait l'objet du chapitre III. *C. Chevalley* (Princeton, N. J.).

***Miller, George Abram. Collected Works, v. 2.** University of Illinois Press, Urbana, 1939. xi+537 pp. $7.50

This volume includes all but 40 of the 147 memoirs published by G. A. Miller during the years from 1900 through 1907, those which constitute his chief contribution to the theory of finite groups during this period, and continues the list of 62 memoirs already published in volume 1. Many of these papers represent special results enumerating the possible finite groups which satisfy given conditions regarding such things as (1) the prime factors which divide the order, (2) the orders of two generating substitutions and their product, (3) the types of subgroups, (4) the degree of a representation as a substitution group. Several papers introduce group-theoretic methods in the treatment of problems in arithmetic, number theory, and trigonometry.

Two papers written especially for this volume are placed at the beginning and the end. The first of these brings up to 1908 a history of the theory of groups, and gives statistical evidence based on the number of printed papers to show the popularity of the subject of finite groups at that time. Three other Bulletin reports on the then recent advances in group theory are included among the memoirs. The last paper (No. 171) enlarges on some "Primary Facts in the History of Mathematics" which had interested the author during this period, making reference to some of the more recent historical researches. Considerable interest is shown here in the ancient and modern thought patterns which encouraged or impeded the development of new mathematical ideas. *J. S. Frame* (Providence, R. I.).

Zappa, Guido. Sulla non-semplicità di alcuni gruppi d'ordine pari. Rend. Sem. Mat. Roma **3**, 54–56 (1939). [MF 510]

If G is a group of even order $2^n(2m+1)$, if there exists an abelian subgroup P of order 2^k which is not contained in an abelian subgroup of order 2^{k+1} so that all the elements not equal to 1 in P are of order 2 and so that the index of the centralizer of P in the normalizer of P is exactly 2, then there exists a normal subgroup of G whose index in G is even. *R. Baer* (Urbana, Ill.).

Miller, G. A. Prime power groups determined by the number of their subgroups. Proc. Nat. Acad. Sci. U. S. A. **25**, 583–586 (1939). [MF 449]

It is shown that every noncyclic group of order p^m except the quaternion group contains at least $(m-1)(p+1)$ proper subgroups. In general, that is, for $m>2$ and $p^m>16$, there are just two groups of the given order which attain this minimum. If this minimum is exceeded, it is by a multiple of p^2. For a given prime p the excess is exactly p^2 for just one abelian group of order p^4 and for just two abelian groups of order p^3 and p^4, when p is odd, but only for the two groups of order 16 when $p=2$. *J. S. Frame.*

Cartan, Henri. Un théorème sur les groupes ordonnés. Bull. Sci. Math. **63**, 201–205 (1939). [MF 364]

It is known that a commutative ordered archimedean group is isomorphic to a subgroup of the additive group of real numbers. The author shows that the hypothesis of commutativity is redundant. In fact, any group with archimedean ordering is essentially cyclic, and so abelian. *G. Birkhoff* (Cambridge, Mass.).

Sinkov, A. A note on a paper by J. A. Todd. Bull. Amer. Math. Soc. **45**, 762–765 (1939). [MF 341]

The defining relations given by J. A. Todd [J. London Math. Soc. **2**, 103–107 (1936)] for the abstract groups $LF(2,p^n)$ may be considerably simplified. The $n+2$ gener-

ators may be reduced to two, of periods 2 and 3, except in the single case $p^n=9$. Explicit defining relations in terms of these two generators are given for the groups $LF(2,2^n)$, $n=3,4,5$, and $LF(2,3^3)$, but are not found in the general case. A simple set of defining relations for $LF(2,p^n)$ in terms of three (when $p=2$) or four (when $p>2$) dependent generators is obtained to replace Todd's definition.

J. S. Frame (Providence, R. I.).

Manning, W. A. On transitive groups that contain certain transitive subgroups. Bull. Amer. Math. Soc. **45**, 783–791 (1939). [MF 344]

This paper contains an elementary and direct proof, that is, a proof without any reference to the theory of representations, of H. Wielandt's theorem [cf. Math. Z. **40**, 582 (1935)] that a group G of permutations is imprimitive if its degree is not a prime number, if it is simply transitive, and if it contains a regular abelian subgroup of the same degree as G, one of whose Sylow-subgroups is cyclic.

R. Baer (Urbana, Ill.).

Magnus, Wilhelm. On a theorem of Marshall Hall. Ann. of Math. **40**, 764–768 (1939). [MF 321]

The author gives a new proof of a theorem of the reviewer's on the extension of an "operator free" abelian group by a finite group [Ann. of Math. (2) **39**, 220–234 (1938)]. In addition he gives a metric representation of the extension which is of interest in itself. The proof is brief but depends on the deep theorem of Schreier on the number of generators of a (necessarily free) subgroup of a free group. *M. Hall* (New Haven, Conn.).

Zappa, Guido. Sull'ampliamento degli automorfismi. Rend. Sem. Mat. Roma **3**, 133–138 (1939). [MF 506]

If N is a normal subgroup of the group G, if the set S of elements in G generates $G \bmod N$, and if α is an automorphism of N, then a necessary and sufficient condition for the existence of an automorphism of G, inducing α in N, is the existence of elements g' corresponding to the elements g in S with the following properties: (i) the elements g' generate $G \bmod N$; (ii) if the element g in S induces the automorphism γ in N, then g' induces the automorphism $\alpha\gamma\alpha^{-1}$ in N; (iii) if the elements in S satisfy the relation $f(\cdots g\cdots)=x$ for some element x in N, then $f(\cdots g'\cdots)=\alpha(x)$. [Cf. R. Baer, Math. Z. **38**, 375–416 (1933); M. Hall, Ann. of Math. **39**, 220–234 (1938); A. M. Turing, Compositio Math. **5**, 357–367 (1938).] *R. Baer* (Urbana, Ill.).

Jacobson, N. Structure and automorphisms of semi-simple Lie groups in the large. Ann. of Math. **40**, 755–763 (1939). [MF 320]

The paper discusses Lie groups from a topologico-algebraic point of view, with the emphasis on semi-simple groups. A topological group is called "topologically semi-simple" when all its closed commutative invariant subgroups are discrete. It is shown that, for Lie groups, this is equivalent to the assertion that the Lie algebra is semi-simple in the usual sense. Moreover, the quotient-group of such a Lie group over its (discrete) central is a direct product of simple Lie groups. Turning to simple Lie groups (groups whose closed proper invariant subgroups are discrete), the author gives an elegant new classification, determining the groups of their automorphisms in all but a finite number of exceptional cases. In all cases, the group of inner automorphisms has finite index. The proofs of all of the above facts and others depend directly on previous work of the author and others. *G. Birkhoff* (Cambridge, Mass.).

✱Pontrjagin, L. Topological Groups. Translated from the Russian by Emma Lehmer. Princeton Mathematical Series, v. 2. Princeton University Press, Princeton, 1939. ix+299 pp. $4.00.

This book is a lucid and comprehensive account of the present state of the subject. The notion of topological group arose out of that of Lie group by abstracting its algebraic, topological and continuity properties and ignoring the analytical features (that is, a topological group is a topological space whose points form the elements of a group in such a way that the group operations are continuous in the topology). The major development of the subject took place during the past ten years, chiefly through the efforts of van Dantzig, von Neumann and Pontrjagin. The result has been the casting of new light on the older theory of Lie groups and the solution of some of its fundamental problems. Chapters I and II develop the elements of group theory and topology requisite for the sequel. Chapter III introduces the subject proper with an axiomatic development. Here the fundamental operations (homomorphism, factor group, direct product) are defined and the basic theorems concerning them are proved. Structure theorems for locally compact 0-dimensional groups are obtained. Chapter IV deals with the integration (mean-value) of complex-valued continuous functions on a compact group, the subsequent theory of integral equations, and concludes with the fundamental theorem of von Neumann on the existence of a complete system of representations of a compact group. Chapter V contains the author's principal contribution to the subject: the notion of character group of a locally compact, separable, abelian group. The character group is shown to be a group of the same type and the relation is proved to be symmetric. Thus the structure of such a group can be studied in terms of that of its character group. This is of particular importance for compact groups since their character groups are discrete and countable. As applications topological-algebraic characterizations of the abelian Lie groups (toral and vector groups) are obtained. The chapter concludes with the author's theorem that the only locally compact, separable, connected, topological fields are the real numbers, the complex numbers and the quaternion field. Chapter VI is a brief introduction to Lie groups with emphasis on local groups, canonical coordinates, and the uniqueness of differentiable coordinates. Chapter VII presents von Neumann's results on the approximation of compact groups by Lie groups and the characterization of compact Lie groups as compact, locally euclidean groups. Chapter VIII presents the notions of fundamental group, covering space, and universal covering group. The last chapter (IX) continues the study of Lie groups by means of their infinitesimal groups (Lie algebras). A brief sketch is given (without proofs) of the classification of complex and compact Lie groups. Likewise included is the proof that a local Lie group extends to an entire group. The author does not attempt to achieve in every case the most general results. The most notable omissions in this connection are the Haar integral, the almost periodic functions of von Neumann and the theorems of Auerbach and Weil. Likewise the discussion of character groups is restricted to the separable case (second axiom of countability) though much extends to the non-separable as shown by van Kampen. The book is very clearly written and abounds in paragraphs descriptive of the objects in mind and the extent in which they are achieved. A very notable and helpful feature is the inclusion of seventy-five examples.

N. E. Steenrod (Chicago, Ill.).

TOPOLOGY

Stopher, E. C., Jr. Point set operators and their interrelations. Bull. Amer. Math. Soc. **45**, 758–762 (1939). [MF 340]

First the author takes the operator dA, which is undefined and concerning which there are assumed two postulates usually assumed in connection with derived sets. He then defines nine other operators, most of them involving in their definition this undefined one. They give the author the interior, complement, isolated points, frontier, etc., of the set A. He first considers all second order operators of which there will be 100. A table is furnished giving all reductions that the author has been able to make either by replacing a second order operator by a single one or by expressing it in terms of operators preceding the given combination in the table of definitions. He also exhibits a table showing the reductions which occur if the set upon which the second order operators are assumed to act is restricted to being S, the set of all points. In Part II of his paper, the author again has his operators act upon a general set, but introduces a third postulate which makes the set S dense in itself and considers all product operators that can be generated by his set when the frontier and border operators are not included. He shows that these operators can be expressed in a certain canonical form. *J. R. Kline.*

Wilder, R. L. Property S_n. Amer. J. Math. **61**, 823–832 (1939). [MF 280]

In 1920, Sierpinski introduced the notion of a set having property S, that is, for every $\epsilon>0$, the set is the sum of a finite number of continua of diameter less than ϵ. R. L. Moore modified this property by requiring that, for every $\epsilon>0$, the set be the sum of a finite number of connected sets of diameter ϵ. The modified property is stronger than local connectivity, weaker than uniform local connectivity, and is a necessary and sufficient condition that a simply connected plane domain have a continuous curve for its boundary. The author introduces an extended S property as follows: pair (U,V) means $U\supset V$, $h^i(U,V)$ denotes the maximum number of i-cycles of V independent with respect to bounding on U; a set of pairs covers M if for every p of M there is a pair (U,V) such that p belongs to V. Denoting by $\sum_U P$ the set theoretic sum of all U's in the set of pairs P and abbreviating $h^i(\sum_U P, \sum_V P)$ as $h^i(P)$, then a set has property S_n if, for every $\epsilon>0$, it is the sum of a finite set P of pairs, each of diameter less than ϵ, such that, for every subset P' of P, $h^n(P')$ is finite. He shows that S_0 is equivalent to S and that S_n is stronger than local n-connectedness. Letting lc^n mean local i connectedness for all $i\leqq n$ and S_j^k imply the property $S_i(j\leqq i\leqq k)$, he shows that, if, in E_n, D is a simply $(n-1)$ connected domain and has property S_0^{n-2}, then its boundary is a Peano continuum. If M is a uniform lc^n whose closure is compact, then M has property S_0^n. *J. R. Kline.*

Jones, F. B. Concerning certain linear abstract spaces and simple continuous curves. Bull. Amer. Math. Soc. **45**, 623–628 (1939). [MF 61]

The author first studies a Hausdorff space M which is linear, that is, one in which for every point P of R, a domain with respect to M, there exists in R a domain with respect to M which contains P and has at most two boundary points with respect to M. He shows that a connected linear Hausdorff space is normal, contains no triod and is locally compact; that it be metric it is necessary and sufficient that it be completely separable. He then applies his results to Moore spaces, showing that every nondegenerate linear connected subset of a Moore space is homeomorphic with a simple continuous curve and that a nondegenerate locally connected complete Moore space is linear if and only if it contains no simple triod. This generalizes previous work of R. L. Moore, R. L. Wilder and the author principally by all omission of compactness requirements. *J. R. Kline.*

Halperin, Israel. Dimensionality in reducible geometries. Ann. of Math. **40**, 581–599 (1939). [MF 98]

If, in a continuous geometry, one has two decompositions of the same element, then for each decomposition there exists a direct refinement which is equivalent to a refinement of the other. It is then possible to show that one has a dimension function even when the condition of irreducibility on the geometry is replaced by a condition of irreducibility under a group of automorphisms, that is, no element except 0 or 1, invariant under every automorphism of the group, has a unique complement. Except at one point, the first mentioned result permits the use of the usual process for showing the existence of a dimension function. Examples of reducible geometries to which this discussion applies are direct finite sums of isomorphic irreducible geometries. Also the notion of "measurability" and a metric for functions whose domain is a fixed measurable set of finite measure and whose values are in a given continuous geometry are introduced. If the given geometry is irreducible the set of such functions forms a reducible continuous geometry to which the preceding discussion applies. *F. J. Murray* (New York, N. Y.).

Whyburn, G. T. Non-alternating interior retracting transformations. Ann. of Math. **40**, 914–921 (1939). [MF 330]

The paper contains two existence theorems on non-alternating interior retracting transformations. It is shown that a Peano space M may be retracted into an arc ab by such a transformation if and only if M is a cyclic chain from a to b, and it may be retracted into a simple closed curve J if and only if M is cyclicly connected and not unicoherent about J. The paper also points out that any interior transformation defined on an arc ab may be extended interiorly to any Peano space which contains ab and is a cyclic chain from a to b, and a similar result for the simple closed curve. *W. L. Ayres* (Ann Arbor, Mich.).

Whyburn, G. T. On irreducibility of transformations. Amer. J. Math. **61**, 820–822 (1939). [MF 279]

A continuous transformation $T(A)=B$ on a compact set A is strongly irreducible (no proper closed subset of A maps into all of B) if and only if the set of points where T is (1-1) is dense in A. If A is a compact semi-locally-connected continuum, then T is irreducible (no proper subcontinuum of A maps into all of B) if and only if the set of points where T is (1-1) is dense in the non-cut points of A. Every irreducible T is strongly irreducible if and only if the non-cut points of A are dense in A. The results are proved by means of the real-valued function $f(x)=\delta(T^{-1}T(x))$ defined on the points x of A. *W. L. Ayres.*

Ayres, W. L. On transformations having periodic properties. Fund. Math. **33**, 95–103 (1939). [MF 442]

A study is made of four properties of single valued transformations of the type $f(X)=X$. The properties P_i $(i=1, 2, 3, 4)$ are, respectively, periodicity, pointwise periodicity, almost periodicity and pointwise almost periodicity. After remarking that P_1 implies the other three and P_2 or

P_3 implies P_4, it is shown that each of the first three, but not P_4, implies that f is (1-1). Then f is restricted to a homeomorphism on a compact Peano continuum X and having property P_4, and a study is made of the action of f on X relative to the cyclic elements of X. It is shown, for example, that if C_1 and C_2 are invariant cyclic elements, the cyclic chain (C_1, C_2) is invariant and, indeed, every individual cyclic element in the chain (C_1, C_2) is invariant. Furthermore, there always exists at least one invariant cyclic element, and the sum I_f of all invariant cyclic elements is a Peano continuum made up of cyclic elements of X (thus is an "A-set"). Under f the components of $X-I_f$ are permuted in a definite manner. If C_1 and C_2 are distinct cyclic elements of X and $f(C_1)=C_2$, the cyclic chain (C_1, C_2) contains one and only one invariant cyclic element. In the concluding section examples are given showing that although relative to the cyclic elements of X, all four of the properties yield the same results, in general the properties are really distinct; and theorems are proven showing that for certain special spaces X, the weaker properties imply the stronger ones; for example, if X is a finite linear graph having a branch point, any continuous pointwise almost periodic homeomorphism on X is necessarily periodic.

G. T. Whyburn (Charlottesville, Va.).

Brouwer, L. E. J. Zum Triangulationsproblem. Nederl. Akad. Wetensch., Proc. **42**, 701–706 (1939). [MF 418]

The author proves that a closed differentiable manifold can always be triangulated. Aside from intuitionistic considerations, this theorem had been proved by Cairns in 1935 [Bull. Amer. Math. Soc. **41**, 529 (1935)].

W. Hurewicz (Chapel Hill, N. C.).

Pietsch, Hans. Geodätische Approximation einer topologischen Triangulation. Deutsche Math. **4**, 583–589 (1939). [MF 527]

The author proves that a given triangulation of a two-dimensional topological complex which is at the same time a Riemannian surface can be deformed into an equivalent triangulation whose arcs are broken geodesics. He assumes the existence of regular geodesic polar coordinates around each point extending a distance bounded away from zero, and he devotes most of the paper to a painstaking demonstration that the approximations by broken geodesics to the sides of the original triangulation (approximations obtained in the way that has been standard since the work of Darboux) need not have intersections except at the vertices of the triangulation, which remain fixed during the approximation and deformation. *C. B. Tompkins.*

ANALYSIS

✱Enzyklopädie der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen. Band I. Algebra und Zahlentheorie. Teil 1. A. Grundlagen. B. Algebra. Heft 2. B. G. Teubner, Leipzig, 1939. 114 pp. RM 6.

This part of the encyclopaedia contains three articles. The first (I1, 3), written by **F. Bachmann**, is "Aufbau des Zahlensystems." It begins with a statement of Peano's axioms and certain other methods of introducing the integers. There follows an account of how the rationals may be constructed as pairs of integers, and from them the set of all real numbers by the methods of fundamental sequences, sections, and some others. Complex numbers are also introduced and there is a brief discussion of the topic of absolute values in fields and the bearing of this question on the fields introduced.

The second article (I1, 4), entitled "Darstellung der reellen Zahlen durch Grenzprozesse," is by **K. Knopp.** It contains an account of real numbers obtained by such devices as the following: upper and lower bounds of sets of numbers, infinite products, infinite series, continued fractions and several others. The nearness of approximation is considered and there is a section on the question of the irrationality of the limits.

The third article (I1, 5) is by **E. Kamke** and has the title "Allgemeine Mengenlehre." This is a treatment of the general theory of sets. The fundamental operations of sum, intersection, limit, and Suslin scheme are first introduced. Next comes the definition of equivalence of sets, Bernstein's theorem, cardinal numbers and their arithmetic. The author then gives an account of some of the work on the foundations of the subject. The last part of the article deals with order, the well ordering theorem, ordinal numbers, transfinite induction, the set of alephs, and some of the applications. *D. Montgomery* (Northampton, Mass.).

✱Gillespie, R. P. Integration. Oliver and Boyd, Edinburgh, 1939. viii+126 pp. 4/6.

The first four chapters are devoted to an elementary account of integration, no attempt being made to be rigorous. The following chapters penetrate deeper into the classic Riemannian theory of integration.

Contents: Introduction. Integration of elementary functions. Multiple integrals. Curvilinear and surface integrals. The Riemann integral. Infinite integrals. The Riemann double integrals.

✱Sokolnikoff, Ivan S. Advanced Calculus. McGraw-Hill Book Co., New York, 1939. x+446 pp. $4.00.

✱Rutherford, D. E. Vector Methods. Oliver and Boyd, Edinburgh, 1939. viii+127 pp. 4/6.

The object of this book is to provide, on the one hand, a clear account of the abstract theory, and, on the other hand, a brief but broad survey of the applications.

Contents: Vector algebra. Differential geometry. Applications to mechanics. The vector operator "∇." Potential theory. Hydrodynamics. Laplace equation. Four-dimensional vectors.

✱Taylor, James Henry. Vector Analysis. Prentice-Hall, Inc., New York, 1939. ix+180 pp. $2.85.

This book is intended to give an introduction to vector and tensor analysis for those students who have studied a first course in the calculus. The development of the subject is an axiomatic one. Thus, after a discussion of translations in three space in a general affine coordinate system a vector is defined as "a quantity whose instances (states) admit of a one-to-one reciprocal continuous correspondence with a set of translations in space." The laws governing operations with vectors are then postulated and discussed. This procedure enables the author to introduce many ideas which are fundamental in algebra and geometry but which are not usually discussed in elementary texts on vector analysis.

Chapter I, "Algebra of Vectors," contains in addition to the material usually found in other texts the following: the Cauchy-Schwarz inequality, the construction of a

unitary orthogonal system of base vectors from three linearly independent ones, and an example which shows that a quantity described by a directed line segment is not necessarily a vector.

Chapter II, "Differential Calculus of Vectors," contains in addition to the usual material an application of vector analysis to curves and surfaces based mainly on the work of Blaschke. Chapter III, "Integral Calculus of Vectors," contains the usual material developed in an intuitive manner only. Chapter IV, "Introduction to Tensor Analysis," seems to make a sharp break with the earlier chapters because the author has not previously emphasized the relation between a vector and its components in a particular coordinate system. The component notation is not wholly adopted in this chapter; for example, the formula for the covariant derivative of a vector is written in terms of base vectors. This chapter includes a brief discussion of the Riemann curvature tensor.

Throughout the book there are problems some of which are applications of the text and some of which are extensions of the theory. Because the book is written for mathematics students, most of the applications are made to mathematics instead of other fields. A large bibliography is included in the text and references are given so that the student may supplement the book by additional reading.

A. H. Taub (Seattle, Wash.).

Theory of Functions of Real Variables, Theory of Measure and Integration

Roger, Frédéric. Sur l'extension à l'ordre n des théorèmes de M. Denjoy sur les nombres dérivés du premier ordre. C. R. Acad. Sci. Paris **209**, 11–14 (1939). [MF 217]

The author has previously [Acta Math. **69**, 99–133 (1937)] established generalizations of Denjoy's theorems on derivates of a function of one real variable [J. Math. Pures Appl. (7) **1**, 105–240 (1915)]. In this note he announces that, by similar methods, he has obtained extensions of Denjoy's theorems to derivates of order higher than the first. From a function $f(x)$ and $(n+1)$ points $x_0, x_1, \cdots, x_n$, he forms the ratio of two determinants

$$Q_{n,k}(f: x_0, x_1, \cdots, x_n)=n!\frac{\begin{vmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^{n-1} & f(x_0) \\ 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} & f(x_1) \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} & f(x_n) \end{vmatrix}}{\begin{vmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^{n} \\ 1 & x_1 & x_1^2 & \cdots & x_1^{n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & x_n & x_n^2 & \cdots & x_n^{n} \end{vmatrix}}$$

and defines the right (left) extreme derivates (n,k) (of order n and rank k) of $f(x)$ at x_0 as the extreme limits of $Q_{n,k}$ when $x_1, \cdots, x_n$ tend to x_0 independently, but with k points more on the right (left) of x_0 than on the left (right). If a right and a left extreme derivate are equal their common value is a "presque" derivative, if all four are equal, a "generalized" derivative. The author states [Theorem 1] that the four derivates (n,k) satisfy the Denjoy relations by being, almost everywhere, equal if finite. Furthermore, except in a set of measure zero, the existence of a derivative (n,k) (presque or generalized) implies (i) the existence (under certain restrictions on n' and k') of a derivative (n',k'), where $n'<n$, (ii) the existence, when the values of $f(x)$ on a set of measure zero are ignored, of ordinary derivatives of $f(x)$ up to the order $[(n+k)/2-1]$, the last of these derivatives possessing a presque or ordinary derivative equal to the derivative $[(n+k)/2, (n+k)/2]$ of $f(x)$.

The author cites, as an example of the application of his results to the geometry of sets of points, the theorem that a convex set in three dimensions possesses almost everywhere an osculating paraboloid.

U. S. Haslam-Jones.

Dickinson, D. R. On the derivation of discontinuous functions. Proc. Cambridge Philos. Soc. **35**, 373–381 (1939). [MF 542]

Let $f(x)$ be a real function of the real variable x, P any point on the graph of $f(x)$, and l a ray from P making an angle $\theta(-\pi<\theta\leqq\pi)$ with the positive direction of the x-axis; θ is a derivate direction at the point P if the ray l meets the graph of $f(x)$ in a set of points with limit point P. Let E be the set of rays from P, mE the linear measure of the set in which the rays of E meet the unit circle with center at P. If $f(x)$ is continuous, it follows from a well-known theorem due to Denjoy on the possible disposition of derivates that for almost all values of x one of the following is true: (1) $f'(x)$ exists. (2) Except for values $\pi/2$, $-\pi/2$, either (i) every value of θ is a derivate direction of $f(x)$, or (ii) every value of θ in a certain interval of measure π is a derivate direction of $f(x)$. The author points out that one would expect similar results if in (1) $f'(x)$ is replaced by some kind of generalized derivative, and if in (2) a set of values of θ of zero measure is excepted. He then shows that this expectation is not realized by constructing a function $\varphi(x)$ which is: (a) continuous for almost all x; (b) not differentiable in any generalized sense for almost all x; (c) if P is a point on the graph of $\varphi(x)$, then at P the set of derivate directions is of measure zero. The function φ is obtained as the limit of a sequence of step functions.

R. L. Jeffery (Wolfville, N. S.).

Krzyżański, Mirosław. Sur l'extension de l'opération intégrale de Denjoy aux fonctions de deux variables. Bull. Sém. Math. Univ. Wilno **2**, 41–51 (1939). [MF 116]

The main purpose of this paper is to extend to two dimensions the so-called descriptive definition of the Denjoy integral which is given in terms of the integral function. Any rectangle will be tacitly assumed to have sides parallel to the axes. Let $F(R)$ be an additive rectangle function defined on a rectangle R_0, let E be a subset of R_0 and let R_E denote the smallest rectangle containing E. The function $F(R)$ is said to be absolutely continuous on E if for each $\epsilon>0$ there exists a $\delta>0$ such that $\sum_{i=1}^{n}|F(R_i)|<\epsilon$ for each finite set of non-overlapping rectangles with total measure less than δ, provided further that each R_i is in R_E and contains a point of E. If $F(R)$ is absolutely continuous on E, it follows that $F(R)$ is absolutely continuous on the closure of E. If $F(R)$ is absolutely continuous on a closed set E, there exist two rectangle functions $\Phi(R)$ and $\Psi(R)$ such that (1) $F(R)=\Phi(R)+\Psi(R)$, (2) $\Phi(R)$ is absolutely continuous on R_0 with $\Phi(R)=0$ if R contains no points of E in its interior, (3) $\Psi(R)$ is absolutely continuous on E, and (4) the Banach derivative $D\Psi$ $(=\lim_{R\to P}\Psi(R)/m(R)$, R being a square containing P) exists and equals zero almost everywhere on E so that $DF=D\Phi$ almost everywhere on E.

$F(R)$ is said to be A.C.G. on R_0 if R_0 is the sum of a denumerable number of sets E_n on each of which $F(R)$ is absolutely continuous. It is shown that, if $F(R)$ is A.C.G. on R_0, DF exists almost everywhere on R_0; if $DF\geqq0$ almost everywhere on R_0, then $F(R)\geqq0$ for each R. A function

$f(x, y)$ is said to be integrable D on R_0 if it coincides almost everywhere with the Banach derivative of a function $F(R)$ which is A.C.G. The usual elementary theorems for the Denjoy integral are extended to two dimensions. The reviewer noted an error in the proof of theorem IV, part 1, but was able to change the developments in part 1 so as to obtain the principal conclusions stated in that part.

C. B. Morrey, Jr. (Berkeley, Calif.).

Adams, C. R. and Clarkson, J. A. A correction to "Properties of functions $f(x,y)$ of bounded variation." Trans. Amer. Math. Soc. **46**, 468 (1939). [MF 476]

Concerning Trans. Amer. Math. Soc. **36**, 711–730 (1934).

Price, G. Baley. A class of monotone functions. Amer. J. Math. **61**, 941–946 (1939). [MF 289]

A real function $x(s,t)$ defined in a closed convex set C of the euclidean plane is linear-monotone if it is monotone on every line segment with extremities on the boundary of C. The significant result established is that C may be covered by a simply ordered set E of such segments S, no two of which intersect in the interior of C, in such a manner that: (a) The discontinuities of $x(s,t)$ interior to C all lie on a countable subset E_1 of segments of E. These are simple discontinuities. The saltus of $x(s,t)$ across each segment (of discontinuities) is constant at all interior points of the segment. (b) At all interior points of a segment S (of continuity) $\epsilon E-E_1$, $x(s,t)$ has a constant value $u(S)$. If S is (geometrically) between S_1 and S_2, then $u(S)$ is between $u(S_1)$ and $u(S_2)$. (c) A similar order relation holds for segments of discontinuity with an appropriate definition of betweenness for the function values on such segments. It is shown that a countable set of discontinuities of a second type (helical) may exist on the boundary of C, thus indicating that the region of definition of $x(s,t)$ may have a natural boundary.

H. E. Bray (Houston, Tex.).

Löwner, Karl. Grundzüge einer Inhaltslehre im Hilbertschen Raume. Ann. of Math. **40**, 816–833 (1939). [MF 325]

The theory of any (Lebesgue or Jordan) measure in Hilbert space has to overcome this difficulty: In n-dimensional Euclidean space the ratio of the measures of two spheres of radii r_1 and r_2 is $(r_1/r_2)^n$. If $r_1>r_2>0$, then this tends to infinity if $n\to\infty$. Hence, if all spheres of positive radius are to have finite and positive measure, then the limiting cases for $n\to\infty$ (and Hilbert space is such a limiting case) are hardly manageable. The above paper proposes to overcome this difficulty by defining measure in Hilbert space not as a real number, but rather as an element of a non-archimedean domain of quantities. Heuristically the idea may be put as follows: Denote the (desired) measure of the (infinitesimal) spherical shell between the radii r and $r+dr$ by $\epsilon_r dr$. In n-dimensional space ϵ_r is a real number: $\epsilon_r = C_n r^{n-1}$ (C_n being a finite, positive constant, depending on n); in Hilbert space treat the ϵ_r as independent hypercomplex units in such a manner that for $r_1>r_2>0$ the ϵ_{r_1} is deemed to be greater than any real numerical multiple of ϵ_{r_2}.

The present paper determines a wide class of bodies (sets) in Hilbert space which can be converted into a combination of such (infinitesimal) spherical shells by some intuitively measure conserving operation, specifically, by a suitably defined extension of Cavalieri's principle. If a body is thus found to be equivalent to a certain superposition of (infinitesimal) spherical shells, the one between the radii r and $r+dr$ being represented with a relative weight $w(r)$, then the measure or volume $\int_0^\infty w(r)\epsilon_r dr$ will be adscribed to it. Considering what was said above about the ϵ_r we can equally well use the function $w(r)$ itself, defining the volume $w_1(r)$ to be greater than the volume $w_2(r)$ if, for some r_0, $w_1(r_0)>w_2(r_0)$ and also $w_1(r)\geqq w_2(r)$ for all $r>r_0$. (This heuristic definition must be amplified, of course, so as to disregard unessential modifications of $w(r)$ on r-sets of measure 0.) It is impossible to describe these processes within the available space with more details or rigorously. We will only indicate the class of sets in Hilbert space for which this extended Cavalieri process works. Any finite dimensional subspace (not necessarily containing the origin) $\mathfrak{A}$ of the Hilbert space $\mathfrak{H}$ is called an axis. Any set $\mathfrak{S}$ in $\mathfrak{H}$ which is transformed into itself by every rotation (that is, one to one, isometric mapping of $\mathfrak{H}$ on $\mathfrak{H}$) and which leaves all points of $\mathfrak{A}$ fixed, is called a rotational body with the axis $\mathfrak{A}$. Observe that in a finite dimensional $\mathfrak{H}$ all sets $\mathfrak{S}$ would be rotational bodies, since we could choose $\mathfrak{A}=\mathfrak{H}$. Since $\mathfrak{H}$ is Hilbert space, however, always $\mathfrak{A}\neq\mathfrak{H}$. Indeed, the rotational bodies play in this case a rôle which is comparable to that of polyhedra in a finite dimensional space. Certain types of limits of rotational bodies are called rotative bodies and these form the above mentioned class of sets. The paper also contains a discussion of many remarkable features of the geometry of rotational and of rotative bodies. A continuation of the work is announced.

J. von Neumann (Princeton, N. J.).

Theory of Functions of Complex Variables

Kasner, Edward and De Cicco, John. The derivative circular congruence-representation of a polygenic function. Amer. J. Math. **61**, 995–1003 (1939). [MF 297]

The derivative dw/dz of a polygenic function $w=\phi(x,y)+i\psi(x,y)$ may be written $H(x,y)+iK(x,y)+e^{-2i\theta}[h(x,y)+k(x,y)]$, where $dy/dx=\tan\theta$. The authors here investigate when a clock congruence or set of four functions H, K, h, k may belong to the derivative of a polygenic function. Similarly, they study when a circle congruence or set of three functions H, K and R, where $R^2=h^2+k^2$, may belong to such a derivative. In each case a suitable set of partial differential equations are found to be necessary and sufficient conditions, and a geometric interpretation of the results is formulated.

P. Franklin (Cambridge, Mass.).

Keldysh, M. V. Conformal mappings of multiply connected domains on canonical domains. Uspekhi Matem. Nauk **6**, 90–119 (1939). (Russian) [MF 396]

A survey of the recent development of the field.

Wolf, František. An extension of the Phragmén-Lindelöf theorem. J. London Math. Soc. **14**, 208–216 (1939). [MF 245]

The following extension of the Phragmén-Lindelöf theorem is given in this note: Let $f(z)$ be regular in $\mathfrak{J}z>0$, let its boundary values on $\mathfrak{J}z=0$ be in absolute value less than 1, and let there exist for every $\epsilon>0$ an $R(\epsilon)$ such that $|f(z)|=|f(re^{i\theta})|\leqq\exp(r\epsilon e^{\psi(\theta)})$ for $r>R(\epsilon)$, where $\psi(\theta)$ is L-integrable in $(-\pi/2, \pi/2)$. Then $|f(z)|\leqq 1$ in $\mathfrak{J}z>0$.

R. Nevanlinna's method for proving the Phragmén-Lindelöf theorem (that is, estimating $\log^+|f(z)|$ by use of the "Poisson integral" for a semicircle) cannot be applied here because the dominant for $\log^+|f(z)|$ is not integrable

on the circular part of the boundary. The author overcomes this difficulty by replacing the semicircle by a certain Jordan arc Γ; Γ is constructed in such a manner that an expression essentially equivalent to the normal derivative of the Green's function of the region obtained by him is $O(e^{-\psi(\theta)})$ on Γ. As an application the following extension of another theorem of Phragmén and Lindelöf [see Titchmarsh, Theory of Functions, 182–184] is mentioned: Let $f(z)$, $z=re^{i\theta}$, be regular in the angle $\alpha<\theta<\beta$, $r>0$ and let

$$h_\rho(\theta)=\overline{\lim_{r\to\infty}}\frac{\log|f(re^{i\theta})|}{r^\rho}.$$

If $h_\rho(\theta)\leqq e^{-\psi(\theta)}$, $\psi(\theta)$ L-integrable in (α,β), then $h_\rho(\theta)$ is continuous in $\alpha<\theta<\beta$. *S. E. Warschawski.*

Iyer, V. Ganapathy. The Phragmén-Lindelöf theorem in the critical angle. J. London Math. Soc. 14, 286–292 (1939). [MF 435]

The author proves several theorems of which the following is an example: Let $f(z)$ be regular for $|\mathrm{am}\, z|\leqq\pi/2\rho$. Let $\{\lambda_n\}$ be a measurable sequence of order ρ and density $D>0$ and let the index of condensation of $\{\lambda_n{}^\rho\}$ be finite. Let

$$\lim_{r\to\infty}\frac{\log M(r,f)}{r^\rho}<\infty,$$

where $M(r,f)=\max|f(re^{i\theta})|\,(|\theta|\leqq\pi/2\rho)$, and let

$$|f(re^{\pm\pi i/2\rho})|\leqq A,\qquad \overline{\lim_{n\to\infty}}\frac{1}{\lambda_n{}^\rho}\log|f(\lambda_n)|\leqq 0.$$

Then $|f(z)|\leqq A$ for $|\mathrm{am}\, z|\leqq\pi/2\rho$. *N. Levinson.*

Golusin, G. M. Interior problems of the theory of univalent functions. Uspekhi Matem. Nauk 6, 26–89 (1939). (Russian) [MF 395]

A detailed survey of the recent development of the field, containing a unified exposition and numerous bibliographical references.

Maitland, B. J. On analytic functions bounded at a double sequence of points. Proc. London Math. Soc. 45, 440–457 (1939). [MF 388]

Let $\{z_{m,n}\}$, $m,n=0,\pm1,\cdots$, be a double sequence satisfying $m-1\leqq\Re z_{m,n}\leqq m$, $n-1\leqq\Im z_{m,n}\leqq n$. An integral function of order two and type less than $\pi/8$ which is bounded over $\{z_{m,n}\}$ must be a constant. Let $\{z_{m,n}\}$ satisfy the additional condition that members are a distance greater than $\rho>0$ apart. An integral function $f(z)$ of order two and type less than $\pi/2$ which is bounded over $\{z_{m,n}\}$ must be a constant; $f(z)$ may be expanded by means of the Lagrange interpolation formula in terms of

$$F(z)=e^{-az^2}z\Pi'\left\{\left(1-\frac{z}{z_{m,n}}\right)\exp\left(\frac{z}{z_{m,n}}+\frac{z^2}{2z_{m,n}^2}\right)\right\},$$

where $2a=\sum'1/z_{m,n}^2$. It is shown that any integer power of $f(z)$ may be expanded in the same way, which leads to a contradiction. *A. C. Schaeffer* (Palo Alto, Calif.).

Obrechkoff, Nikola. Sur les zéros de quelques classes de fonctions. Comment. Math. Helv. 12, 66–70 (1939). [MF 499]

(1) Let $f(x)$ be a polynomial with real and positive roots or, more generally, an integral function which is the limit of such polynomials. Let $p>0$, $q>0$, $y(x)=x^pf(x^q)$. Then $f_n(x)=y^{(n)}(x^{1/q})$ vanishes only for $x\geqq0$ if $n<p+1$; this is true for $n<p+q+1$ if p is integral, and for all n if both p and q are integral. If $f(x)$ is a polynomial with real roots or an integral function which is the limit of such polynomials, then the function $f_n(x)$ defined above can have only real roots. (2) Based on a theorem of Pólya, the following is proved: Let $F(x)$ be integrable in $[-1,+1]$, and let $\int_{-1}^{+1}F(x)e^{ixz}dx$ be the limit of polynomials with only real roots. Also, let $f(x)$ be a polynomial whose roots lie in the strip $c_1\leqq\Re x\leqq c_2$. Then the same holds for the polynomial $\int_{-1}^{+1}F(x)f(z+x)dx$. *G. Szegö.*

Hornich, Hans. Über transzendente Integrale erster Gattung. Monatsh. Math. Phys. 47, 380–387 (1939). [MF 68]

Aufstellung der Normalintegrale erster Gattung auf den Riemannschen Flächen der Funktionen $y^2=g(x)$, wo $g(x)$ eine ganze Funktion mit nur einfachen reellen Nullstellen ist [vergl. Monatsh. Math. Phys. 40, 241–282 (1933); 42, 377–388 (1935)]. *W. W. Rogosinski.*

Reuter, G. Ein Interpolationsproblem. Ark. Mat., Astr. Fys. 26, no. 18, 8 pp. (1939). [MF 308]

The author considers the class of analytic functions $f(z)$ regular and bounded in the strip $|\Im z|\leqq D$ which assume preassigned values ω_ν at the points $z=\nu$, $\nu=0,\pm1,\pm2,\cdots$, and for which

$$N(f)=\int_{-\infty}^{+\infty}|f(x+iD)|^2dx+\int_{-\infty}^{+\infty}|f(x-iD)|^2dx$$

is finite. He shows that the necessary and sufficient condition for the existence of such functions is the convergence of $\sum|\omega_\nu|^2$. Assuming this, the minimum of the expression $N(f)$ is attained for a function of the form

$$f(z)=\sum_{\nu=-\infty}^{+\infty}\frac{c_\nu}{\cosh\frac{\pi}{4D}(z-\nu)},$$

where $\sum|c_\nu|^2$ is convergent; the constants c_ν are determined from the ω_ν by solving certain finite systems of linear equations and by a proper limiting process. The theory of bilinear forms of an infinite number of variables is used. *G. Szegö* (Stanford University, Calif.).

Montel, Paul. Sur les suites de fonctions non bornées dans leur ensemble. Bull. Soc. Math. France 67, 42–55 (1939). [MF 444]

Several sets of conditions are obtained which imply that a family of functions $f(z)$ be normal, when the functions do not satisfy the standard condition of normality, that they be bounded in their set. These conditions are given in terms of the maximum modulus, the means of Hardy, the maximum of the real part of $f(z)$, the area covered by the map $w=f(z)$, and the maximum modulus of the pth derivative $f^{(p)}(z)$. A typical result is the following: A family of functions $f(z)$ holomorphic in a closed domain (D) is normal if, for all points interior to (D),

$$|f(z)|\geqq\delta M,\qquad 0<\delta\leqq1,$$

where M denotes the maximum of $|f(z)|$ in (D). *E. F. Beckenbach* (Houston, Tex.).

Vignaux, Juan-Carlos. Sur les familles normales de fonctions holomorphes (α). C. R. Acad. Sci. Paris 209, 147–149 (1939). [MF 225]

Following D. Pompeiu the author calls $f(z)$, $z=x+iy$,

"holomorphe (α)" (h. α.) in a region D if its areal derivative $\varphi(z)$ exists at every point $z_0 \subset D$ ($\varphi(z_0) = \lim (1/\sigma)\int_C f(z)dz$, where C is a rectifiable closed Jordan curve whose interior contains z_0 and has the area σ, and where C is being deformed continuously into z_0). Two theorems on functions h. α. of the type of Montel's theorems on normal families of analytic functions are stated: (1) A family of functions $f(z)$ which are h. α. in D and uniformly bounded on the rectifiable boundary curve Γ of D and whose areal derivatives $\varphi(z)$ are uniformly bounded in D, is normal in D. (2) If $f_n(z)$, $n=1, 2, \cdots$, are h. α. in D and converge uniformly on Γ, and if the corresponding $\varphi_n(z)$ converge uniformly in D, the $f_n(z)$ converge uniformly in D. Various extensions to other classes of polygenic functions are mentioned. *S. E. Warschawski* (St. Louis, Mo.).

Vignaux, Juan-Carlos. Sur les séries simples et doubles asymptotiques de Dirichlet. C. R. Acad. Sci. Paris **209**, 84–87 (1939). [MF 220]

The author is concerned with Dirichlet series (ordinary, general, double series, and irregular power series) which are asymptotic to a given analytic function in the sense of Poincaré. He states uniqueness of the representation and discusses how such series behave under the operations of addition, multiplication, division, differentiation, and integration. [See J. F. Ritt, Amer. J. Math. **50**, 73–86 (1928), and A. Ostrowski, Math. Z. **37**, 98–133 (1933), where even more general questions are solved.] Applications to the Borel-Carleman theorem in the theory of quasi-analytic functions are indicated. *E. Hille* (New Haven, Conn.).

Badescu, Radu. Sopra una certa serie di Laurent di due variabili. Boll. Un. Mat. Ital. **1**, 314–322 (1939). [MF 572]

The author considers the functional equation $\varphi(z) = \lambda\varphi(\alpha z)$, where α is fixed, $|\alpha|<1$, and λ is a complex parameter, and he investigates properties of solutions representable in the form $\sum_{n=-\infty}^{\infty}\lambda^n q(\alpha^n z)$, where $q(z)$ is analytic in some region of the z-plane. He shows that a solution of this form cannot have a pole of any order at the origin in the λ-plane, for z belonging to a certain (arbitrarily small) region, without being identically zero. Also such a solution cannot be holomorphic in z near $z=0$. These properties furnish an answer to a question raised by Popovici on the analytic character of such solutions [Mathematica, Cluj **9**, 201 (1935)]. These results enable one to draw consequences in the case of the nonhomogeneous equation $\varphi(z) - \lambda\varphi(\alpha z) = \psi(z)$. *W. T. Martin* (Cambridge, Mass.).

Taylor, A. E. A theorem concerning analytic continuation for functions of several complex variables. Ann. of Math. **40**, 855–861 (1939). [MF 327]

The author gives a new proof of theorems of Hartogs and Levi, which state that every single valued analytic function of n complex variables, $n>1$, which is regular on the connected boundary $\mathfrak{t}^{2n-1}$ of a bounded domain $\mathfrak{T}^{2n}$ can be extended analytically to a single-valued and analytic function throughout $\overline{\mathfrak{T}}^{2n}$. (Analytic can be replaced here and in the following by meromorphic.) [Cf. also A. B. Brown, Duke Math. J. **2**, 20 (1936).] The author uses the known theorem to the effect that every function of n complex variables which is regular in the neighborhood $[\mathfrak{U}^{2n}(P) - \mathfrak{S}^{2n}\cdot\mathfrak{U}^{2n}(P)]$ of a point P lying on the boundary of a hypersphere $\mathfrak{S}^{2n}$ can be analytically extended in $\mathfrak{S}^{2n}\cdot\mathfrak{U}^{2n}(P)$. The essential novelty of his proof consists in introducing the concept of "attainable sphere," that is, a sphere $\mathfrak{S}^{2n}$ such that $\mathfrak{S}^{2n}\cdot\mathfrak{T}^{2n}$ is not empty, that f is regular in $\mathfrak{G}^{2n}(\mathfrak{G}^{2n} \supset \overline{\mathfrak{T}}^{2n} - \mathfrak{S}^{2n}\cdot\mathfrak{T}^{2n})$ and which possesses certain properties which cannot be given in detail here. The author shows that if $\mathfrak{S}^{2n}(Q,r)$ (a sphere of radius r and center Q), $Q \not\subset \overline{\mathfrak{T}}^{2n}$, is an attainable sphere, then there exists an $\epsilon>0$ such that $\mathfrak{S}^{2n}(Q,r-\epsilon)$ is also an attainable sphere. Furthermore he proves that this shrinking process can always be continued until the sphere $\mathfrak{S}^{2n}(Q, r)$ has common points with $\mathfrak{T}^{2n}$. *S. Bergmann* (Cambridge, Mass.).

Fuks, B. A. On the invariant Riemannian metrics in the theory of pseudo-conformal mappings and its applications. Uspekhi Matem. Nauk **6**, 251–286 (1939). (Russian) [MF 398]

A survey and unified exposition of the recent literature.

Theory of Series

Fort, Tomlinson. The Euler-Maclaurin summation formula. Bull. Amer. Math. Soc. **45**, 748–754 (1939). [MF 338]

Extension of the well-known summation formula to multiple sums. *O. Szász* (Cincinnati, Ohio).

Bradshaw, J. W. Modified series. Amer. Math. Monthly **46**, 486–492 (1939). [MF 360]

The author modifies a given convergent series of constants so as to increase the rapidity of convergence. The modification consists in adding to the partial sum S_n a number $b_n \to 0$ such that the modified sequence of partial sums $S_n + b_n$ approaches a limit more rapidly than S_n. This can always be accomplished if the given series is one of the forms $\sum(-1)^n/x_n$ or $\sum 1/x_n$, where x_n is a polynomial in n. If the method is repeated, the successive modifications will in some cases be the convergents of a continued fraction; in such a case the remainder of the series after n terms is represented by the continued fraction. Several particular cases are discussed in detail, among which are $\sum(-1)^n/n = -\log 2$, $\sum 1/n^2$, $\sum(-1)^n/(2n-1) = -\pi/4$. *P. W. Ketchum* (Urbana, Ill.).

Kuttner, B. Some theorems on Riesz and Cesàro sums. Proc. London Math. Soc. **45**, 398–409 (1939). [MF 385]

Let $\kappa \geqq 0$, and let $A^{(\kappa)}(\omega)$ and $A_n^{(\kappa)}$ be defined, in terms of a series $\sum a_n$, as usual for the Riesz and Cesàro means (R, n, κ) and (C, κ). The well-known fact that (R, n, κ) and (C, κ) are equivalent implies (and is implied by) the fact that, for each series $\sum a_n$, the estimates $A^{(\kappa)}(\omega) = o(\omega)$ and $A_n^{(\kappa)} = o(n^\kappa)$ both hold or both fail to hold. This paper shows that (i) "$A^{(\kappa)}(\omega) \geqq 0$ for $\omega>0$" implies (ii) "$A_n^{(\kappa)} \geqq 0$ for $n>0$" when $0 \leqq \kappa \leqq 1$, but that (i) does not imply (ii) when $\kappa>1$; and that (ii) implies (i) when κ is an integer but that (ii) does not imply (i) when κ is not an integer. The same conclusions hold when (i) and (ii) are replaced respectively by the unilateral conditions (i′) "$A^{(\kappa)}(\omega) = O_L(\omega^r)$ for $\omega>0$" and (ii′) "$A_n^{(\kappa)} = O_L(n^r)$ for $n>0$," in which r is a positive number. The proofs make use of and apply to the "function-to-function" integral transformations which are analogues of the Riesz and Cesàro "series-to-function" and "series-to-sequence" transformations. *R. P. Agnew* (Ithaca, N. Y.).

Agnew, Ralph P. Properties of generalized definitions of limit. Bull. Amer. Math. Soc. **45**, 689–730 (1939). [MF 333]

The theory of matrix transformations which serve to

associate limits with divergent sequences and series has been extensively studied. The theory of kernel transformations of the type

$$y(s)=\int_0^\infty K(s,t)x(t)dt,$$

which may be utilized to associate with a function that does not tend to a limit a related function which possesses this property, is considerably less developed. The present expository paper, based on an address delivered before the American Mathematical Society, furnishes a clear and concise presentation of the fundamental theory of kernel transformations as it exists at present. The point of view adopted for the exposition is novel and affords a more general basis for the theory than is found in the previous literature. Instead of postulating any particular definition of the integral the author prescribes six properties of the integral which are needed for the development of the theory. These properties are common to the integrals as defined by Riemann and Lebesgue. The author then characterizes the class of functions $K(s,t)$ that will be used as kernels and proceeds to the setting up of criteria which determine what properties the transformation in question has or fails to have. Among the important questions discussed are those that relate to the regularity of the transformation and to further useful properties, in the field of generalized limits, possessed by regular transformations of certain types.

C. N. Moore (Cincinnati, Ohio).

Meyer-König, Werner. Limitierungsumkehrsätze mit Lückenbedingungen. II. Math. Z. **45**, 479–494 (1939). [MF 403]

The author proves several theorems in which he shows that if $s_n=a_1+\cdots+a_n$ is Euler summable to a limit s, if the a_n satisfy certain Tauberian restrictions over an infinite sequence of large intervals in n, and if the s_n satisfy certain restrictions on their order of growth, then a certain subsequence of $\{s_n\}$ converges to s. *N. Levinson.*

Sundaram, S. Minakshi. On generalised Tauberian theorems. Math. Z. **45**, 495–506 (1939). [MF 404]

The author deals with relations between the behavior of

$$F(\sigma)=\sigma\int_0^\infty \varphi(\sigma t)s(t)dt$$

as $\sigma\to 0$ and that of $s(t)$ as $t\to\infty$. Here $\varphi(t)$ is positive, differentiable, ultimately monotone, and the following integrals exist:

$$\int_0^\infty \varphi(t)dt=1,\qquad \int_0[1-\psi(t)]t^{-1}dt,\qquad \int^\infty \psi(t)t^{-1}dt,$$

where $\psi(t)=\int_t^\infty \varphi(u)du$. Assuming, for instance, that $s(t)$ is differentiable and $\lim\inf_{t\to\infty} ts'(t)\geqq 0$, while $F(\sigma)=O(1)$ as $\sigma\to 0$, he shows that $\operatorname{osc}_{\sigma\to 0} F(\sigma)=\operatorname{osc}_{t\to\infty} s(t)$. Applications to Dirichlet series. *E. Hille* (New Haven, Conn.).

Knopp, Konrad. Limitierungs-Umkehrsätze für Doppelfolgen. Math. Z. **45**, 573–589 (1939). [MF 410]

Let σ_{mn} denote the C_1 transform of the sequence s_{mn} of partial sums of a double series $\sum u_{mn}$. Two double sequences η_{mn} and η'_{mn} which are double-sequence analogues of the Kronecker sequence $\delta_n=(a_1+2a_2+\cdots+na_n)/(n+1)$ are defined and it is shown that, if $\sigma_{mn}\to s$, s_{mn} is bounded, $\eta_{mn}\to 0$, and $\eta'_{mn}\to 0$, then $s_{mn}\to s$. Let s_{mn} be real; let $\kappa,\lambda>1$; let $w_{m,n}(\kappa,\lambda)$ be the maximum for $m\leqq\mu\leqq\kappa m$, $n\leqq\nu\leqq\lambda n$ of the numbers $(s_{\mu\nu}-s_{mn})$; and let $w(\kappa,\lambda)$ be the superior limit as $m,n\to\infty$ of $w_{m,n}(\kappa,\lambda)$. If $\sigma_{mn}\to s$ and $w(\kappa,\lambda)\to 0$ as $\kappa\to 1+$, $\lambda\to 1+$, then $s_n\to s$. This unilateral Tauberian theorem implies that if $\sigma_{mn}\to s$ and $(m^2+n^2)a_{mn}<M$, then $s_n\to s$. For Abel's power series method A of summability, the following Tauberian theorem is proved. If $\sum a_{mn}x^m y^n$ converges over the square $0<x,y<1$ to a function $f(x,y)$ which is bounded over the square, if $(m^2+n^2)|a_{mn}|<M$, and if $f(x,y)\to s$, then $s_{mn}\to s$. It is shown in the course of the proof of the last theorem that A and C_1 are equivalent over the set of bounded sequences s_{mn}. *R. P. Agnew* (Ithaca, N Y.).

Fourier Series and Integrals, Theory of Approximation

Grünwald, Géza. Zur Summabilitätstheorie der Fourierschen Doppelreihe. Proc. Cambridge Philos. Soc. **35**, 343–350 (1939). [MF 539]

Let $F(\xi,\eta)$ be integrable over the square $0\leqq\xi,\eta\leqq 2\pi$ and periodic (of period 2π) in ξ and in η. Let (x,y) be a point where $F(\xi,\eta)$ is continuous, and $\sum_{m,n=0}^\infty A_{mn}$ the double Fourier series expansion of $F(\xi,\eta)$ at (x,y). Let $s_{nn}=\sum_{i,j=0}^n A_{ij}$. Then the sequence $\{s_{nn}\}$ is $(C,1)$ summable to $F(x,y)$. This is an extension of a recent result of Fejér [Proc. Cambridge Philos. Soc. **34**, 503–509 (1938)] who proved that $\{s_{nn}\}$ is $(C,3)$ summable to $F(x,y)$. The author points out that the result can be extended to any number of dimensions. *J. D. Tamarkin* (Providence, R. I.).

Cooper, J. L. B. The absolute Cesàro summability of Fourier integrals. Proc. London Math. Soc. **45**, 425–439 (1939). [MF 387]

Bosanquet [Proc. London Math. Soc. (2) **41**, 517–528 (1936)] obtained very general results connecting the absolute summability of Fourier series by Cesàro means with the bounded variation of Cesàro means of the function in the neighborhood of a given point. Cooper establishes analogous results for Fourier integrals.

O. Szász (Cincinnati, Ohio).

Fox, C. A class of Fourier kernels. J. London Math. Soc. **14**, 278–281 (1939). [MF 433]

The function $m_1(x)$, defined by

$$m_1(x)=(2/\pi)^{\frac{1}{2}}\int_0^\infty (1-\cos xt)J_{2n+\frac{1}{2}}(t)\frac{dt}{t^{\frac{1}{2}}},$$

and expressible in terms of hypergeometric functions, is shown to be the kernel $k_1(x)$ of a general transform [for the theory of these transforms, see E. C. Titchmarsh: Introduction to the Theory of Fourier Integrals, 1937, chap. 8]. When $n=0$, the m-transform of $f(x)$ is $f(1/x)/x$; when $n\to\infty$, $n^{\frac{1}{2}}m_1(x/n)\to\pi^{-\frac{1}{2}}(1-\cos 2x)$. Thus the m-transform is a link between the simplest of all general transforms and the Fourier sine transform. Similar statements apply to the function $p_1(x)$ defined by

$$p_1(x)=(2/\pi)^{\frac{1}{2}}\int_0^\infty \sin(xt)J_{2n+\frac{1}{2}}(t)\frac{dt}{t};$$

the limiting case is the Fourier cosine transform.

R. P. Boas (Durham, N. C.).

Erdös, Paul. On a family of symmetric Bernoulli convolutions. Amer. J. Math. 61, 974–976 (1939). [MF 293]

Let a family of distribution functions $\lambda(x; a)$ be defined by the formula

$$\int_{-\infty}^{+\infty} e^{iux} d_x\lambda(x; a) = \prod_{n=0}^{\infty} \cos(a^n u)$$

(that is, $\lambda(x; a)$ is a convolution of the infinitely many Bernoulli distributions $\beta(a^{-n}x)$). In addition to the known results of Wintner and Kershner [Amer. J. Math. 57, 576–577 and 837 (1935)], it is proved that, if $\alpha > 1$ is an algebraic integer of degree m and such that all of its conjugates $\alpha_2, \cdots, \alpha_m$ satisfy the inequality $|\alpha_j| < 1$, $\lambda(x; \alpha^{-1})$ is purely singular. For example, α may be the "Fibonacci" number $\frac{1}{2}(5^{1/2}-1)$ or the positive root of the equation $a^3+a^2-1=0$.
M. Kac (Ithaca, N. Y.).

Boas, R. P., Jr. A trigonometric moment problem. J. London Math. Soc. 14, 242–244 (1939). [MF 427]

Let $\{\lambda_n\}$, $n=0, \pm 1, \pm 2, \cdots$ be a sequence of real numbers such that $|\lambda_n - n| < L < \infty$. Let $f(t) \epsilon L^2(-\pi, \pi)$ and

$$a_n = \int_{-\pi}^{\pi} e^{i\lambda_n t} f(t)dt.$$

Then there exists a constant A_2 depending only on L and such that

$$\sum |a_n|^2 \leqq A_2 \int_{-\pi}^{\pi} |f(t)|^2 dt.$$

Conversely, if $L < \pi^2$ and $\{a_n\}$ is a sequence of constants such that $\sum |a_n|^2 = K < \infty$, then there exists a function $f(t) \epsilon L^2(-\pi, \pi)$ such that

$$a_n = \int_{-\pi}^{\pi} e^{i\lambda_n t} f(t)dt \quad \text{and} \quad \int_{-\pi}^{\pi} |f|^2 dt \leqq A_1 K,$$

where A_1 depends only on L. These results are analogous to those obtained by Paley and Wiener [Fourier Transforms in the Complex Domain, Amer. Math. Soc. Coll. Publ. 19, 114–115 (1934)], but are proved by a different method, based on a lemma due to Paley and Wiener [ibid., 108] and on a result of F. Riesz concerning the generalized problem of moments. *J. D. Tamarkin* (Providence, R. I.).

Cossar, J. On conjugate functions. Proc. London Math. Soc. 45, 369–381 (1939). [MF 250]

A theorem of M. Riesz states that if $f(x) \epsilon L^p(-\infty, \infty)$, $p > 1$, then there is a $g(x) \epsilon L^p(-\infty, \infty)$ such that

$$g(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(t)}{t-x} dt,$$

where the integral is a Cauchy principal value at $t=x$; and

$$(*) \qquad \int_{-\infty}^{\infty} |g|^p dx \leqq A_p \int_{-\infty}^{\infty} |f|^p dx,$$

where A_p is a number depending on p but not on f. Moreover, the conjugate of g is $-f$. As Riesz has shown, the complete theorem can be deduced (by "convexity" and "conjugacy" arguments) from the special case where p is an even integer. The author gives a new, strictly "real variable" proof of this special case. He first considers the case $p=2$ (when $A_p=1$ and there is equality in (*)) and then deduces the case $p=2k$ ($k=2, 3, \cdots$) from this. The central idea of the first part is to define $g(x)$ first by

$$g(x) = \lim_{\delta \to 0} \frac{1}{\pi} \left(\int_{-\infty}^{x-\delta} + \int_{x+\delta}^{\infty} \right) \frac{f(t)dt}{t-x},$$

and identify it with the ordinary limit by the use of a singular integral. For the second part, it is sufficient to establish the result for real functions which have bounded derivatives and vanish outside a finite interval; the result for more general functions is readily obtained by approximation. If $f(x)$ is such a special function, $g(x)$ is its conjugate, k is an integer, and $(f+ig)^k = R_k(f, g) + iI_k(f, g)$, the author shows that $R_k(f, g)$ belongs to L^2 and has $I_k(f, g)$ as its conjugate. The result, for $p=2k$, is obtained in a simple and elegant way from this lemma. *R. P. Boas, Jr.*

Macaulay-Owen, P. Parseval's theorem for Hankel transforms. Proc. London Math. Soc. 45, 458–474 (1939). [MF 389]

The author proves Parseval's formula

$$\int_0^{\infty} f(x)g(x)dx = \int_0^{\infty} F(x)G(x)dx,$$

where $F(x)$ and $G(x)$ are the Hankel transforms of $f(x)$ and $g(x)$, respectively, subject to the conditions (i) $f(x)$ Lebesgue integrable in $(0, \infty)$ and (ii) $g(x)$ of bounded variation in $(0, \infty)$ and such that $g(x)$ tends to zero as x tends to infinity. *A. C. Offord* (Cambridge, England).

Feller, Willy. Completely monotone functions and sequences. Duke Math. J. 5, 661–674 (1939). [MF 182]

Let

$$(1) \qquad f(x) = \int_0^{\infty} e^{-xt} dF(t),$$

where $F(t)$ is non-decreasing for $t \geqq 0$, the integral converging for $x > 0$. A fundamental theorem states that a function $f(x)$ allows of such a representation (1) if and only if $f(x)$ has for $x > 0$ derivatives of any order such that $(-1)^n f^{(n)}(x) \geqq 0$ ($n=0, 1, 2, \cdots; x > 0$). The author establishes the following inversion formula for (1):

$$F(t) = \lim_{\eta \to \infty} \sum_{n=0}^{[t\eta]} \frac{(-\eta)^n}{n!} f^{(n)}(\eta)$$

in any continuity point of $F(t)$, furnishing at the same time a very brief and elegant independent proof of the fundamental theorem stated above. The same idea allows the author to give greatly simplified proofs of theorems of Widder [Trans. Amer. Math. Soc. 33 (1931)] expressing conditions on $f(x)$ in order that (1) $F(t)$ be of bounded variation in every finite interval, the integral (1) converging absolutely for $x > 0$; (2) $F(t)$ be the integral of its uniformly bounded derivative. Finally, the author considers a sequence $\{a_n\}$ ($a_n \geqq 0$; $n=0, 1, \cdots$) which is completely monotone with respect to another sequence $\{x_n\}$ ($0 \leqq x_0 < x_1 < \cdots$; $x_n \to \infty$) in the sense of Hausdorff [Math. Z. 9, 282 (1921)] and solves the interpolation problem $f(x_n) = a_n$ ($n=1, 2, \cdots$), in case $\sum x_n^{-1} = \infty$, by a function $f(x)$ which is completely monotone for $x \geqq x_0$ and explicitly given by Newton's interpolation series. This seems to be essentially Hausdorff's method [loc. cit.]. The monotone function $F(t)$ corresponding by (1) to the solution $f(x)$ of the interpolation problem is also directly expressed in terms of the sequences $\{a_n\}$ and $\{x_n\}$, a fact which, according to the author, is of importance in the statistical control of telephone service in large

cities. A footnote mentions a recent note by J. Dubourdieu [C. R. Acad. Sci. Paris **206**, 556–557 (1938)] as related to the subject and methods of this paper.

I. J. Schoenberg (Waterville, Me.).

Hartman, Philip. On Dirichlet series involving random coefficients. Amer. J. Math. **61**, 955–964 (1939). [MF 291]

The author considers the family of Dirichlet series $D(s, \theta)=\sum\varphi_n(\theta)a_n n^{-s}$, where $\{a_n\}$ is a given sequence and $\{\varphi_n(\theta)\}$ is the Rademacher orthogonal system. There exist numbers $\bar{C}\leqq\bar{U}\leqq\bar{A}$ such that the values of θ for which $D(s,\theta)$ is convergent, uniformly convergent and absolutely convergent for $\sigma>\bar{C}$, $\bar{U}$, and $\bar{A}$, respectively, form sets of measure one. Here $0\leqq\bar{U}-\bar{C}\leqq\bar{A}-\bar{C}\leqq\frac{1}{2}$ can be proved by simple means and the main object of the paper is to show that $\bar{A}-\bar{U}\leqq\frac{1}{2}$ is the best inequality of its kind. This also gives a new proof of the Bohr-Toeplitz theorem according to which the abscissas of absolute and of uniform convergence may differ by as much as $\frac{1}{2}$ [first proof by Bohnenblust and Hille, Ann. of Math. (2) **32**, 600–622 (1931)]. The proof is based on an interesting theorem on almost periodic functions. Let $\sum|c_n|^2<\infty$, let $\Lambda_n=k_{n,1}\lambda_1+\cdots+k_{n,j}\lambda_j$, $j=j_n$, where the λ_n are linearly independent with respect to rational numbers and the k's are integers such that $|k_{n,1}|+\cdots+|k_{n,j}|\leqq N$, a fixed integer, then the almost periodic function of class B_2 defined by $\sum c_n\varphi_n(\theta)\exp(i\Lambda_n t)$ is uniformly almost periodic for a set of values of θ of measure 1, provided in addition $\sum|c_n|^2 j_n \log^{2+\epsilon} j_n<\infty$ for every $\epsilon>0$. This theorem is used to prove that $\bar{A}-\bar{U}$ may reach its maximum value $\frac{1}{2}-1/(2N)$ for Dirichlet series in which every n is the product of N primes, from which the desired result readily follows. [In a different direction the referee has proved that in a suitable metric the Dirichlet series for which $\bar{A}-\bar{U}<\frac{1}{2}$ form a set of the first category.]

E. Hille (New Haven, Conn.).

Lewitan, B. Über eine Verallgemeinerung der stetigen fastperiodischen Funktionen von H. Bohr. Ann. of Math. **40**, 805–815 (1939). [MF 324]

The author proves theorems similar to those given in earlier papers [C. R. Acad. Sci. U.R.S.S. **17**, 287–290; **19**, 447–450 and Commun. Inst. Sci. Math. et Méc. Univ. Kharkoff (4) **15**, 3–35]. The new theorems differ from the earlier ones simply in the class of generalized almost periodic functions to which they apply. According to the new definition, a function $f(x)$ is N-almost periodic if to every pair $\epsilon>0$, $N<+\infty$ there corresponds an almost periodic set of integers $\tau_n=\tau_n(\epsilon, N)=-\tau_{-n}$ such that $|f(x+\tau_n)-f(x)|<\epsilon$ whenever $|x|<N$. This definition is simpler and neater than the earlier ones, and the proofs of theorems based on it are shorter. *R. H. Cameron* (Cambridge, Mass.).

van Kampen, E. R. On the asymptotic distribution of a uniformly almost periodic function. Amer. J. Math. **61**, 729–732 (1939). [MF 32]

For a given real, almost periodic function $f(t)$, $-\infty<t<+\infty$, let A_x; B_x; C_x denote the t-sets on which $f(t)\leqq x$; $f(t)<x$; $f(t)=x$. According to Wintner, every $f(t)$ has an asymptotic distribution function $\sigma=\sigma(x)$ in the following sense: If x is a continuity point of $\sigma(x)$, then any of the three sets A_x; B_x; C_x has a relative measure, represented by $\sigma(x)$; $\sigma(x)$; 0, respectively. An example of Bohr shows that this does not need to hold if x is a discontinuity point of $\sigma(x)$. The author applies the method of condensation of singularities to a suitable modification of this example to construct a function $f(t)$ for which the exceptional set of x-values is a given bounded sequence $x_1, x_2, \cdots$.

B. Jessen (Copenhagen).

Kac, M., van Kampen, E. R. and Wintner, Aurel. On the distribution of the values of real almost periodic functions. Amer. J. Math. **61**, 985–991 (1939). [MF 295]

Let $f(t)=F(\lambda_1 t, \cdots, \lambda_n t)$, where F is real and of period 1 in each argument and has certain "smoothness" properties. The authors show by the theory of geometric probabilities and the Kronecker-Weyl theorem that the asymptotic density $E(\lambda)$ of solutions of $f(t)=a$ can be expressed as an integral connected with the hypersurface H: $F(\theta_1, \cdots, \theta_n)=a$. In particular, when H (or its equivalent) is convex, $E(\lambda)=2(\lambda_1^2+\cdots+\lambda_n^2)^{\frac{1}{2}}S$, where S is the $(n-1)$-volume of the projection of H on the hypersurface orthogonal to the vector λ. The computational value of this formula is shown by examples; and these ideas are applied to $\arg[f(t)+ig(t)]$, and thus to the problem of "mean motion." *R. H. Cameron* (Cambridge, Mass.).

Reijnierse, J. M. Sur la limite d'une suite de polynômes. Nieuw Arch. Wiskde **20**, 39–47 (1939). [MF 569]

In a previous paper [Nieuw Arch. Wiskde **19**, 241–248 (1938)], in generalizing a well-known theorem, the author proved that if $\{D_l\}$, $l=1, 2, \cdots$ is a sequence of domains (open) exterior to each other, and $\{f_l(z)\}$ a sequence of functions such that $f_l(z)$ is holomorphic in D_l, then there exists a sequence of polynomials converging to $f_l(z)$ in D_l. In the present paper the author proves the following extension: Let $\{D_l\}$, $\{f_l(z)\}$ be as before and let the complement of $\sum D_l$ be $\sum E_m$, where the E_m are all closed and exterior to each other. Let $\{g_m(z)\}$ be another sequence of functions such that $g_m(z)$ is holomorphic in E_m. Then there exists a sequence of polynomials converging to $f_l(z)$ in each D_l and to $g_m(z)$ in each E_m with the exception of a suitable set S_m. The sets S_m are explicitly constructed in the paper.

J. D. Tamarkin (Providence, R. I.).

Jackson, Dunham. A new class of orthogonal polynomials. Amer. Math. Monthly **46**, 493–497 (1939). [MF 361]

The author gives a brief discussion of some properties of the sequence of polynomials $\{p_n(x)\}$ orthonormal on $(-1, 1)$, with unit weight-function, satisfying the boundary condition

$$\text{(I)}\qquad p_n(1)=p_n(-1)$$

(this common value is taken later to be equal to 0). The $p_n(x)$, for n even, coincide with the normalized even Legendre polynomials; for n odd, $p_n(x)$ is expressed simply in terms of polynomials $q_n(x)$ such that

$$\int_{-1}^{1}(x^2-x)^2 q_m(x)q_n(x)dx=\delta_{mn}, \qquad m,n=0, 1, 2, \cdots.$$

(The substitution $x^2=y$ reduces $q_n(x)$ directly to a Jacobi polynomial on (0,1). Reviewer.) Various recurrence relations are given for $p_n(x)$, including an analogue to the Darboux Formula. The latter enables the author to treat, in the ordinary manner, the convergence of expansions of functions in series of $p_n(x)$. In closing, the author sketches the case when the boundary condition (I) is replaced by $p_n(-1)=-p_n(+1)$. *J. A. Shohat* (Philadelphia, Pa.).

Bailey, W. N. On Hermite polynomials and associated Legendre functions. J. London Math. Soc. **14**, 281–286 (1939). [MF 434]

The author gives a new proof of certain formulas of

Miss I. W. Busbridge [J. London Math. Soc. 14, 93–97 (1939)] involving products of three or four Hermite polynomials and the kernel e^{-x^2}. Generalizing one of these formulas he proves

$$(1)\qquad \int_{-\infty}^{\infty} e^{-x^2} H_s(x) H_{p_1}(x)\cdots H_{p_r}(x)dx=\sqrt{\pi}\,2^s s!,$$

where $s=p_1+\cdots+p_r$. Noticing that Hermite polynomials are confluent forms of associated Legendre functions, he records a number of analogous formulas for the latter functions which give the corresponding Hermite formulas by a suitable limiting process. The analogue of (1), for instance, becomes

$$\int_{-1}^{1}(1-\mu^2)^{-\frac{1}{2}m(r-1)}P^m_{s+m}(\mu)P^m_{p_1+m}(\mu)\cdots P^m_{p_r+m}(\mu)d\mu$$
$$(2)\qquad =\frac{(s+m)!(s+2m)!}{2^{m(r-1)-1}(2s+2m+1)!}\prod_{t=1}^{r}\frac{(2p_t+2m)!}{p_t!(p_t+m)!}.$$

E. Hille (New Haven, Conn.).

Beretta, L. e Merli, L. Sulla convergenza in media della formula di interpolazione di Hermite. Boll. Un. Mat. Ital. 1, 322–330 (1939). [MF 573]

Let $x_k^{(n)}=\cos(k\pi/(n+1))$, $k=1, 2, \cdots, n$, be the zeros of the Chebyshef polynomial of the second kind, $U_n(x)=(\sin(n+1)z)/\sin z$, $x=\cos z$, $n=1, 2, \cdots$. Let $f(x)$ be continuous in $[-1, 1]$, assuming the values $y_k^{(n)}$ at $x=x_k^{(n)}$; let $y_k'^{(n)}$ be the values of $f'(x)$ at $x=x_k^{(n)}$. Finally, let $X_n(x,f)=X_n(x)$ be the Hermitian polynomial of interpolation of f, that is, the polynomial of degree $(2n-1)$ assuming together with its derivative the values $y_k^{(n)}$, $y_k'^{(n)}$ at $x=x_k^{(n)}$. The authors prove that, under the additional restriction $|y_k'^{(n)}|\leqq\epsilon_n n^{1/2}$, $\epsilon_n\to 0$,

$$\int_{-1}^{1}(1-x^2)^{\frac12}[X_n(x)-f(x)]^2dx\to 0 \quad\text{as}\quad n\to\infty.$$

J. D. Tamarkin (Providence, R. I.).

Favard, J. Sur l'interpolation. Bull. Soc. Math. France 67, 102–113 (1939). [MF 448]

If $\{a_\nu^n\}$ ($\nu=0, 1, \cdots, n+1$; $n=0, 1, 2, \cdots$) is a double sequence of points such that $a_0^n=0$, $a_{n+1}^0=1$, $a_\nu^n<a_{\nu+1}^n$, and the maximum distance between a_ν^n and $a_{\nu+1}^n$ approaches zero as $n\to\infty$, there exist sequences of polynomials $A_\nu^n(x)$ of degree at most n, such that for any $f(x)$ continuous in $0\leqq x\leqq 1$

$$\lim_{n\to\infty}\sum_{\nu=0}^{n}f(a_\nu^n)A_\nu^n(x)=f(x),$$

uniformly in $(0,1)$. This theorem, according to the author, is already known; he gives a very simple proof of it. He also discusses classes of continuous functions for which there exist sequences $\{A_\nu^n(x)\}$ giving rise to a "best" process of approximation. In the second part of the paper (independent of the first), new quasi-analytic classes of functions, generalizing those of S. Bernstein, are defined. Such a class $E_{\{n\}}$ is defined with respect to any infinite closed point set E in $(0,1)$ and any increasing sequence of integers n_k. The function $f(x)$, continuous in (a,b), belongs to $E_{\{n\}}(a,b)$ if for $n=n_k$ there exist polynomials $P_n(x)$ of degree at most n such that

$$\sup_{0\leqq x\leqq 1}|f(x)-P_n(x)|\leqq\frac{M_n}{\varphi_E(n)},\qquad \lim_{n\to\infty}M_n=0,$$

where $\varphi_E(n)$ is a certain function depending only on E, and increasing sufficiently rapidly. A member of a class $E_{\{n\}}$ is determined by the values which it assumes on any set E_1, similar to E, but contained in an arbitrarily small subinterval of (a,b). These quasi-analytic classes reduce to Bernstein's if E is an interval (when $\varphi_E(n)=\rho^n$, $\rho>1$).

R. P. Boas (Durham, N. C.).

Differential Equations

✶Ince, E. L. Integration of Ordinary Differential Equations. Oliver and Boyd, Edinburgh, 1939. viii+148 pp. 4/6.

The object of this book is to provide in a compact form an account of the methods of integrating explicitly the common types of equations, and, in particular, those equations which arise from problems in geometry and applied mathematics. It takes the existence of solutions for granted. The book includes 300 examples and solutions.

Kamke, E. A new proof of Sturm's comparison theorems. Amer. Math. Monthly 46, 417–421 (1939). [MF 203]

Comparison theorems are obtained for the second order, linear differential system

$$y'=P(x)y+Q(x)z,\quad z'=R(x)y+S(x)z,$$

where the coefficients are continuous on $a\leqq x\leqq b$. The theorems are proved through use of a change to polar coordinates in the yz-plane. The classical comparison theorems of Sturm for the single linear, self-adjoint, differential equation of the second order occur as special cases of the theorems of the paper. In connection with the use of the polar coordinate transformation to study second order differential systems, additional reference should be made to a paper by the reviewer [Trans. Amer. Math. Soc. 30, 848–854 (1928). In particular, Theorem VII, page 854, bears on the work of the paper under review.].

W. M. Whyburn (Los Angeles, Calif.).

Rothe, Erich. Asymptotic solution of a boundary value problem. Iowa State Coll. J. Sci. 13, 369–372 (1939). [MF 494]

The author undertakes to prove that, if $y(x,\lambda)$ and $\eta(x)$ are, respectively, solutions of the boundary problems

$$y''+\lambda(y'-y)=\lambda f,\qquad y(0)=y(1)=0,$$
$$\eta'-\eta=f,\qquad \eta(1)=0,$$

with λ positive and $f(x)$ continuous on $0\leqq x\leqq 1$, then

$$\lim_{\lambda\to\infty}y(x,\lambda)=\eta(x)\qquad \text{on } 0<x\leqq 1.$$

The proof is unsatisfactory, since it requires among other things the differentiability of $f(x)$, which is not hypothesized.

R. E. Langer (Madison, Wis.).

McShane, E. J. On the uniqueness of the solutions of differential equations. Bull. Amer. Math. Soc. 45, 755–757 (1939). [MF 339]

This paper gives uniqueness theorems for the differential system:

$$y_i'=f_i(x, y_1, \cdots, y_n),\qquad i=1, 2, \cdots, n,$$

where the functions $f_i(x, y_1, \cdots, y_n)$ are measurable in x on (a,b): $a\leqq x\leqq b$ for fixed $(y_1, \cdots, y_n)$, continuous in $(y_1, \cdots, y_n)$ for fixed x on (a,b), bounded by a summable (in the sense of Lebesgue) function $S(x)$, and satisfy one of the following conditions:

(i) There exists a function $M(x)$ summable on (a, b) such that for all x on (a, b), all $(y_1, \cdots, y_n)$ and $(\eta_1, \cdots, \eta_n)$, the inequality

$$(1)\quad \sum_{i=1}^{n}[f_i(x, y_1+\eta_1, \cdots, y_n+\eta_n) \\ -f_i(x, y_1, \cdots, y_n)]\eta_i \leqq M(x)\sum_{i=1}^{n}\eta_i^2$$

holds.

(ii) To each point $(x_0, y_{10}, \cdots, y_{n0})$ with $a \leqq x \leqq b$ there corresponds a positive number ϵ and a function $M(x)$, summable over an interval (α, β) having $\alpha < x_0 < \beta$, such that inequality (1) holds whenever $\alpha \leqq x \leqq \beta$ and $\sum \eta_i^2 < \epsilon$.

(iii) The same condition as (ii) with the left-hand side of (1) replaced by its negative.

(iv) The same as (ii) with the left-hand side of (1) replaced by its absolute value.

(v) The same as (ii) with inequality (1) replaced by

$$\left\{\sum_{i=1}^{n}[f_i(x, y_1+\eta_1, \cdots, y_n+\eta_n)-f_i(x, y_1, \cdots, y_n)]^2\right\}^{\frac{1}{2}} \\ \leqq M(x)\left\{\sum_{i=1}^{n}\eta_i^2\right\}^{\frac{1}{2}}.$$

(vi) $n=1$, and $f_1(x, y_1)$ is a monotonic decreasing function of y_1 for each fixed x.

Through the use of an ingenious method of proof, the theorems are established with little effort. Many of the previously known uniqueness theorems occur as special cases of the ones given in this paper. *W. M. Whyburn.*

Smith, F. C. On the logarithmic solutions of the generalized hypergeometric equation when $p=q+1$. Bull. Amer. Math. Soc. **45**, 629–636 (1939). [MF 62]

In a previous paper [Bull. Amer. Math. Soc. **44**, 429–433 (1938)], the author gave the relations between the non-logarithmic solutions of the differential equation

$$z\prod_{r=1}^{q+1}(\theta+a_r)y=\prod_{s=1}^{q+1}(\theta+c_s-1)y,$$

wherein $\theta=z(d/dz)$, the constants a_r, c_s are any complex quantities $(r=1, 2, \cdots, q+1;\ s=1, 2, \cdots, q)$, and $c_{q+1}=1$. His aim now is to give results for both the logarithmic and non-logarithmic cases. The first case in which there are solutions in the form of series composed partially of terms with logarithmic factors is that in which $c_2-c_1=p_1$, $c_3-c_2=p_2, \cdots, c_m-c_{m-1}=p_{m-1}$, where p_s+1 is in each case a positive integer. It is supposed that no c_s differs from any a_r by an integer. The second case is that in which $a_1-a_2=k_1, a_2-a_3=k_2, \cdots, a_{n-1}-a_n=k_{n-1}$, where k_r+1 is in each case a positive integer. The previous assumption regarding the differences a_r-c_s is again made.

H. Bateman (Pasadena, Calif.).

Erdélyi, A. Integration of a certain system of linear partial differential equations of hypergeometric type. Proc. Roy. Soc. Edinburgh **59**, 224–241 (1939). [MF 252]

The author deals with a certain system of two partial differential equations of the second order for a function of two variables. He proves that every solution can be represented by a certain integral in the integrand of which only a product of functions of one variable occurs. By choosing different paths of integration he obtains eighteen formal solutions, representing ten different solutions. Three sets of fundamental solutions are discussed. The problem of finding a full transformation scheme of the solutions is left to a subsequent paper. *E. Rothe* (Oskaloosa, Iowa).

Bock, Philipp. Einige Integrale aus der Theorie der hypergeometrischen und verwandter Funktionen. Compositio Math. **7**, 123–134 (1939). [MF 372]

Specializing the parameters in some formulae of Erdélyi [J. Indian Math. Soc., New Series **3**, (1939)] author obtains a large number of infinite integrals in the integrand of which products of Whittaker functions and their special cases such as Bessel functions, functions of the parabolic cylinder, Error-function, uncomplete Gamma-function, Bateman's function k, Laguerre and Charlier polynomials occur. *A. Erdélyi* (Edinburgh).

Molenaar, P. G. Ueber Differentialinvarianten zweiter Ordnung der binären kubischen Differentialform. Nederl. Akad. Wetensch., Proc. **42**, 592–600 (1939). [MF 314]

In this continuation of a previous paper [Nederl. Akad. Wetensch., Proc. **42**, 158–166 (1939)], the author determines the differential invariants and covariants of the second order of a binary cubic differential form and in addition determines several syzygies that connect them.

J. Williamson (Baltimore, Md.).

Eger, Max. Sur la jacobienne d'un système de Pfaff. C. R. Acad. Sci. Paris **209**, 82–84 (1939). [MF 219]

Given on an algebraic variety V_n with n-dimensions, without singularities, a system of Pfaff $\omega_\alpha=0$ $(\alpha=1, \cdots, \lambda;\ \lambda\leqq n)$, where the ω_α are rational at a point of V_n, the jacobian variety of the system is the locus of points where the exterior product $[\omega_1 \cdots \omega_\lambda]$ is identically zero. A permissible form ω is one such that each polar hypersurface of ω is an integral of $\omega=0$. If Ω is the intersection of p polar (non-tangent) hypersurfaces of ω and if the $n-1$ spreads tangent to these hypersurfaces at 0 have for equations $x_1=x_2=\cdots=x_p=0$, the author announces that in the neighborhood of 0 we can write a permissible form as follows:

$$(1)\qquad \omega=\frac{1}{L^{k-1}}\left[K_1\frac{dx_1}{x_1}+\cdots+K_p\frac{dx_p}{x_p}+\psi\right],$$

where L is a linear form in x_i, K_i are not all zero at 0, and ψ is a differential form in $dx_{p+1}, \cdots, dx_n$, regular at 0. In view of (1) the intersection of Ω with the jacobian variety of ω and a form F is identical with the jacobian variety of the trace of F on Ω. *E. W. Titt* (Hyattsville, Md.).

Carleman, T. Sur un problème d'unicité pour les systèmes d'équations aux dérivées partielles à deux variables indépendantes. Ark. Mat., Astr. Fys. **26**, no. 17, 9 pp. (1939). [MF 307]

The author proves the following uniqueness theorem: If the functions $z_p(x, y)$ $(p=1, \cdots, n)$ form a system of solutions of the n real differential equations

$$\frac{\partial z_p}{\partial x}+\sum_{q=1}^{n}A_{pq}(x, y)\frac{\partial z_q}{\partial y}+\sum_{q=1}^{n}B_{pq}(x, y)z_q=0,$$

and if all z_p vanish on a segment of the y-axis, then the z_p vanish throughout a whole circle concentric to the segment. The coefficients A_{pq} are supposed to be twice continuously differentiable, and the B_{pq} continuous. Moreover all roots of the characteristic equation $\det(A_{pq}-\lambda\delta_{pq})=0$ shall be simple. Difficulties only arise if some or all of the characteristic roots are imaginary; these are overcome by what

amounts to the solution of the boundary value problem for a particular region. *F. John* (Lexington, Ky.).

Giraud, Georges. Nouvelle méthode pour traiter certains problèmes relatifs aux équations du type elliptique. J. Math. Pures Appl. 18, 111–143 (1939). [MF 379]

The author discusses the existence of a solution of the elliptic partial differential equation

$$F(u)-g^2u=f(x),\quad x=(x_1,\cdots,x_m),\ x\varepsilon D,$$

subject to a boundary condition of the form $\Theta(u)=\varphi(y)$, $y\varepsilon S$, where D is a bounded domain with boundary S. For functions of class C'' in D and C' on $D+S$,

$$\text{(A)}\quad \begin{aligned} F(u)&=\sum_{\alpha,\beta=1}^{m} a_{\alpha\beta}(x)\frac{\partial^2 u}{\partial x_\alpha\partial x_\beta}+\sum_{\alpha=1}^{m} b_\alpha(x)\frac{\partial u}{\partial x_\alpha}+c(x)u(x),\quad x\varepsilon D,\\ \Theta(u)&=\sum_{\alpha,\beta=1}^{m} a_{\alpha\beta}(y)\omega_\beta(y)\frac{\partial u}{\partial x_\alpha}(y)+k(y)u(y),\quad y\varepsilon S;\end{aligned}$$

for functions u, not satisfying the differentiability conditions, $F(u)$ and $\Theta(u)$ are to be replaced by certain generalized operators considered in previous papers. [For F, see Bull. Sci. Math. 56, 248–272, 281–312, 316–352, chap. 1. For Θ, see Ann. Scuola Norm. Super. Pisa (2) 7, 25–71 (1938).] It is assumed that the functions $a_{\alpha,\beta}(x)$ satisfy a Hölder condition on $D+S$, the functions $\omega_\beta(y)$ (the direction cosines of the outer normal) satisfy a similar condition on S, and the functions $b_\alpha(x)$, $c(x)$, and $k(y)$ are continuous.

The author first obtains the explicit form for the Green's function for the case that the $a_{\alpha\beta}$ and ω_β are constants with b_α, c, and $k\equiv 0$. These results are used to construct an "approximate Green's function" $H(x,\xi)$ with the proper differentiability properties (with respect to x) and the proper singularity for $x=\xi$ and which satisfies the condition

$$\begin{aligned} F_x[H(x,\xi)]-g^2H(x,\xi)&=O(|x-\xi|^{h-m}),\\ \Theta_y[H(y,\xi)]&=O(|y-\xi|^{1+h-m}),\quad h>0.\end{aligned}$$

By the substitution

$$u(x)=-\int_D H(x,\xi)\rho(\xi)dV_\xi+\int_S H(x,\eta)\sigma(\eta)dS_\eta,$$

the equations (A) are reduced to a pair of integral equations which may be treated by the Fredholm theory. In the case $c(x)\equiv k(y)\equiv 0$, the exact Green's function is obtained and the general system (A) is reduced to another system of Fredholm integral equations involving the exact Green's function instead of the "approximate Green's function" $H(x,\xi)$. *C. B. Morrey, Jr.* (Berkeley, Calif.).

van den Dungen, F.-H. Une nouvelle définition des partiels. C. R. Acad. Sci. Paris 209, 199–201 (1939). [MF 229]

The lowest eigenvalue ω_1^2 in vibration problems is the minimum of the quotient of two functionals $D[y]:H[y]$. While it is apparently impossible to characterize the higher eigenvalues as minima without reference to the eigenfunctions of lower order, this is possible, for example, for the product and the sum of the first and second eigenvalues:

$$\omega_1^2\omega_2^2=\min\left(\frac{D[y_1]D[y_2]-D^2[y_1,y_2]}{H[y_1]H[y_2]-H^2[y_1,y_2]}\right),$$

$$\omega_1^2+\omega_2^2=\min\left(\frac{D[y_1]H[y_2]+D[y_2]H[y_1]-2D[y_1,y_2]H[y_1,y_2]}{H[y_1]H[y_2]-H^2[y_1,y_2]}\right).$$

K. Friedrichs (New York, N. Y.).

Titchmarsh, E. C. On expansions in eigenfunctions. J. London Math. Soc. 14, 274–278 (1939). [MF 432]

The author considers the boundary value problem

$$(*)\quad \frac{\partial^2\psi}{\partial x^2}+\frac{1}{x}\frac{\partial\psi}{\partial x}=\frac{\partial\psi}{\partial t};\quad \psi(a,t)=0;\quad \psi(x,0)=\psi(x),\quad 0<x<a,$$

where $\psi(x)$ is given. If

$$\psi(x)=\sum_1^\infty A_nJ_0(\alpha_nx),$$

where $\{\alpha_n\}$ is the set of zeros of $J_0(az)=0$, a formal solution of (*) is

$$\sum_1^\infty A_ne^{-\alpha_n^2t}J_0(\alpha_nx).$$

The author shows how this can be justified by using the Fourier transform

$$\Psi(x,w)=(2\pi)^{-\frac12}\int_0^\infty \psi(x,t)e^{iwt}dt,$$

$$\psi(x,t)=(2\pi)^{-\frac12}\int_{ib-\infty}^{ib+\infty}\Psi(x,w)e^{-iwt}dw,$$

where $b>c$ and it is assumed that $\psi(x,t)$ and its derivatives which occur are $O(e^{ct})$ as $t\to\infty$. In the case where $\psi(x,0)=\psi(x)$ for $x>a$, an analogous method leads to a representation of the solution by a repeated integral.

J. D. Tamarkin (Providence, R. I.).

Pleijel, Åke. Propriétés asymptotiques des fonctions fondamentales du problème des vibrations dans un corps élastique. Ark. Mat., Astr. Fys. 26, no. 19, 9 pp. (1939). [MF 306]

In two publications [Skandinaviska Matematikerkongressen, Stockholm 1934, 34 ff., and Ber. Verh. Sächs. Akad. Wiss., Leipzig 1936, 119 ff.] Carleman has applied Tauberian theorems to the theory of the asymptotic behaviour of eigenvalues of partial differential equations of the second order. His methods yield not only the results concerning eigenvalues previously obtained by Weyl and later by Courant, but also give information about the eigenfunctions. The present paper applies Carleman's method to the problem of three dimensional elastic vibrations with fixed boundary, which is a problem of higher order. Following a previous publication by Weyl, the Green tensor of the problem is constructed and appraised and the corresponding integral equation discussed. Then Carleman's method is applied and thus an asymptotic formula for a mean value of the squares of the eigenfunctions is proved, in analogy to a formula by Carleman for the vibrating membrane. Further results concerning asymptotic equal distribution of energy in the body and concerning other problems are announced. *R. Courant* (New York, N. Y.).

Hibbert, Lucien. Propriétés de la fonction harmonique log R définie sur le cercle-unité par des suites particulières de ses valeurs. C. R. Acad. Sci. Paris 209, 287–289 (1939). [MF 232]

Log R is a harmonic function within the unit circle C and V is its conjugate. Boundary values of log R are assigned as follows: On one arc (B,D) log R is a monotone increasing step function, with steps at $A_1, A_2, \cdots$ accumulating at A_∞, where there may also be a step. On the complementary arc, log R is monotone decreasing and continuous. The author proves the theorem: (1) Along a curve

$\log R=\text{const.}$, abutting at A_k, $\lim V(P)=+\infty$. (2) If $f=Re^{iV}$, then $f'\neq 0$ in C, and $f'=0$ exactly once on each arc (A_k, A_{k+1}). A second theorem gives the analogous result for certain sets of boundary values having sets E of points of discontinuity which are perfect and nowhere dense on the circumference of C. *J. W. Green* (Rochester, N. Y.).

Titt, Edwin W. An initial value problem for all hyperbolic partial differential equations of second order with three independent variables. Ann. of Math. **40**, 862–891 (1939). [MF 328]

The author solves a Cauchy problem and proves a uniqueness theorem for his solution in the case of an analytic nonlinear partial differential equation $F=0$ of second order in three independent variables if the equation is "hyperbolic" for the initial data. The term "hyperbolic" is employed in a new sense to cover several essentially different types of initial problems and equations, the scope of which can best be seen by setting forth their simplest linear equivalents:

1. equation $\partial^2z/\partial x^2+\partial^2z/\partial y^2-\partial^2z/\partial t^2=0$, initial surface $t=0$;
2. same equation, initial surface $x=0$;
3. $\partial^2z/\partial x^2-\partial^2z/\partial y^2+\partial z/\partial t=0$, initial surface $x=0$;

with initial values of z and first (normal) derivatives on the initial surface. Cover the initial surface with a family of curves of arc length w and of "space-like" direction with respect to the characteristic cone (relation 2.4) and denote by s a second parameter on the surface. Then the initial conditions (coordinates of surface and initial functions and derivatives up to a certain order) are assumed to be expressible as uniformly convergent power series in w with coefficients continuous in s ("partially analytic in w").

The method used involves derivation of a system of characteristic equations for $F=0$. Upon introduction of proper parameters this system is transformed into a system of the following type for unknown functions $y_\alpha(u, v, w)$, $\alpha=1, 2, \cdots, n$:

$$(\Sigma)\begin{cases}\displaystyle\sum_\alpha a_i^\alpha\frac{\partial y_\alpha}{\partial u}=\sum_\alpha b_i^\alpha\frac{\partial y_\alpha}{\partial w}+\sum_\alpha c_i^\alpha y_\alpha+d_i, & i=1, 2, \cdots, p;\\[2ex] \displaystyle\sum_\alpha a_j^\alpha\frac{\partial y_\alpha}{\partial v}=\sum_\alpha b_j^\alpha\frac{\partial y_\alpha}{\partial w}+\sum_\alpha c_j^\alpha y_\alpha+d_j, & j=p+1, \cdots, n.\end{cases}$$

Here $a_i^\alpha, \cdots, d_j$ are analytic functions of their arguments $u, v, w, y_1, \cdots, y_n$. For (Σ) the author proves by successive approximation the existence of solutions $y_\alpha(u, v, w)$ with initial values given on $u+v=0$ and partially analytic in w. Uniqueness of solutions is proved by only slight alterations in the successive approximations. This uniqueness is then used to infer that the solution of the characteristic system for $F=0$ furnishes a solution of $F=0$. *H. Lewy.*

Churchill, R. V. On the problem of temperatures in a non-homogeneous bar with discontinuous initial temperatures. Amer. J. Math. **61**, 651–664 (1939). [MF 22]

The heat conduction equation

$$U_{xx}(x, t)-q(x)U(x, t)=U_t(x, t), \quad 0<x<1; 0<t,$$

under the boundary conditions

$$U(x, 0+)=F(x), \qquad U(0+, t)=0, \qquad U(1-, t)=0,$$

in which $q(x)\subset C$, $F(x)$ is piecewise continuous, and $F'(x)$ is bounded and Riemann integrable, is transformed by a single (unilateral) Laplace transformation. The transformed problem is solved by means of a Green's function. An indicated inverse transformation shows that the solution of the original problem can be expressed as the classical series solution with Fourier coefficients or left as an inverse Laplace integral. Conditions are given on q, F, and their derivatives under which the solution of the problem is unique. The paper ends with a discussion of the behavior of the solution. *J. L. Barnes* (Merchantville, N. J.).

Vernotte, Pierre. Intégration de l'équation de la convection naturelle. C. R. Acad. Sci. Paris **209**, 19–21 (1939). [MF 218]

Using methods developed by him in a former note [C. R. Acad. Sci. Paris **208**, 1712 (1939)], the author attacks the equation

$$\left(-\frac{\partial\varphi}{\partial z}\frac{\partial}{\partial x}+\frac{\partial\varphi}{\partial x}\frac{\partial}{\partial z}\right)\left[-\frac{\partial\varphi}{\partial z}\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial x\partial z}\frac{\partial\varphi}{\partial x}\right]=\frac{K}{C}\frac{\partial^2}{\partial x^2}\left[-\frac{\partial\varphi}{\partial z}\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial x\partial z}\frac{\partial\varphi}{\partial x}\right],$$

which has application in the theory of the heat conduction through a plane. *E. Rothe* (Oskaloosa, Iowa).

Functional Analysis, Ergodic Theory

Dunford, Nelson and Pettis, B. J. Linear operations among summable functions. Proc. Nat. Acad. Sci. U. S. A. **25**, 544–550 (1939). [MF 271]

A summary of results obtained in the representation of linear operations. Instead of considering functions defined on a Euclidean range, the authors take as a basis a general range S or T of elements, a Borel field $\mathfrak{E}$ of subsets E of S, and a completely additive measure function α on $\mathfrak{E}$. The function α determines the subset $\mathfrak{E}_B$ of sets E_B on which $\alpha(E)<\infty$, and it is assumed that, relative to α, S has a decomposition, that is, there exists a denumerable set of disjoint sets S_i of $\mathfrak{E}_B$ such that $S=\sum_i S_i$. On this basis we then obtain classes $L^q(S)$ of real valued measurable functions φ for which $\int_S|\varphi(s)|^q d\alpha<\infty$, $1\leqq q<\infty$, $L^\infty(S)$ being the class of essentially bounded measurable functions, that is, for which there exists a set E_0 of measure zero ($\alpha(E_0)=0$), such that $\sup|\varphi(s)|$ for s on $S-E_0<\infty$. Similarly, if X is a Banach space, the general integrals on such a space give rise to classes of functions $x(s)$ on S to X; in particular, $A^\infty(S)[X]$ is the class of measurable functions on S to X for which $\|x(s)\|$ is of $L^\infty(S)$. The basic theorem for representations on $L(S)=L^1(S)$ is comparable to the theorem of equivalence between absolutely continuous set functions and indefinite Lebesgue integrals, but the proof is based on a theorem of moments instead of differentiation. X being the adjoint of Y and Y' a separable linear subset of Y, then $x_E(y)$, if it is completely additive and absolutely continuous on each E' of $\mathfrak{E}_B$ for each y of Y', and such that the set function $\sigma(E)=\sup|x_E(y')|$, $\|y'\|=1$, has a finite total variation $v_{E'}$ for each E' of $\mathfrak{E}_B$, implies the existence of a function $x(s)$ on S to X such that $x_E(y)=\int_E x_s(y)d\alpha$ and $v_E=\int_E\|x_s\|d\alpha$. This theorem yields in particular the general linear operation on $L(S)$ to $L(T)$, T being a finite or infinite real interval $-\infty\leqq c<d\leqq+\infty$, via an operation on $L(S)$ to the class $C^*(T)$ on numerical valued functions $v(t)$ of bounded variation over the interior of T such that $v(c+0)=0$, $v(t)=v(t+0)$ and $v(d-0)=v(d)$, which

is the adjoint of the separable space of real valued functions $\gamma(t)$ continuous over T for which $\gamma(c+0)$ and $\gamma(d-0)$ exist finitely, such representation being of the form $d/dt\int_S K(s,t)\varphi(s)d\alpha$ with suitably conditioned K. The paper is also concerned with the representation of separable (transforming a space into a separable set), completely continuous (c.c.) and weakly c.c. (transforming a bounded set into a weakly compact set) operations. A basic theorem is: If S is a bounded or infinite n-dimensional Euclidean space, then a (weakly) c.c. operation U on $L(S)$ to X has the form $U(\varphi)=\int_S x(s)\varphi(s)d\alpha$, where $x(s)$ is an essentially (weakly) compact element of $A^\infty(S)[X]$, such a U taking $L(S)$ for any S into a separable set, and weakly compact sets of $L(S)$ into compact sets of X. If $S=T$, $\alpha=\beta$, and $U(\varphi)$ is weakly c.c. on $L(S)$ to $L(T)$, then there exists a $K(s,t)$ on $S\times T$ such that ess. $\sup_S \int_T |K(s,t)|d\alpha<\infty$ and $U(\varphi)=\int_S K(s,t)\varphi(s)d\alpha$ and U^n is c.c. for $n\geqq 2$. If the sequence of norms $|U^n|$ is bounded, then $\lim_m 1/m\cdot\sum_{n=1}^m U^n$ exists, is c.c., and transforms $L(S)$ into the fixed point space of U, that is, V is an operation of finite dimension, generalizing a mean ergodic theorem of Kakutani and Yosida [Proc. Imp. Acad. Tokyo 14, 292–300 (1938)]. The theorem is applied to a type of Markoff processes which are based on a weakly c.c. operation with norm unity. A theorem for a representation of separable and c.c. operations on $L(S)$ to $L^q(T)$ in the form $\int_S K(s,t)\varphi(s)d\alpha$ is also given. The concluding theorem states that the bilinear form $\int_T x(t)\int_S K(s,t)\varphi(s)d\alpha d\beta$ with suitable conditioned K, and x of $A^\infty(T)[X]$, is a c.c. operation on $L(S)$ to X.

T. H. Hildebrandt (Ann Arbor, Mich.).

Fortet, R. Quelques théorèmes relatifs au calcul du rayon polaire d'une opération linéaire. Rev. Sci. (Rose Illus.) 77, 496–498 (1939). [MF 478]

L'objet de ce mémoire est de donner la démonstration de certains résultats que l'auteur avait brièvement esquissés dans sa thèse [Revista de Ciencias, Lima, 1938]. Le sujet de ces théorèmes est la théorie abstraite des équations intégrales de Fredholm. Soit U une opération linéaire dans un espace $\mathfrak{B}$ dont la résolvante U_λ admet 1 comme pôle de rang infini; soit ce pôle d'ordre 1 et U_1 l'opération principale de U: $U=U_1+U_2$ avec $U_1U_2=U_2U_1=0$, $U_1^2=U_1$. Alors si U_1 est complètement continue, le rayon polaire de U n'est pas moins au rayon polaire de U_2. Un autre théorème affirme que le rayon polaire de l'opération associée $\bar{U}$ de U est égal au rayon polaire de U. *E. R. Lorch* (New York, N. Y.).

Taylor, A. E. The extension of linear functionals. Duke Math. J. 5, 538–547 (1939). [MF 172]

Let P be a projection of a Banach space E on a manifold $\mathfrak{M}$ of E, and let E^* denote the conjugate space of linear functionals defined on E. Properties of the operation T on E^* to $\mathfrak{M}^*$ defined by $T(f(x))=f(x)$ for $x\epsilon\mathfrak{M}$, $f\epsilon E^*$, and the operation A on $\mathfrak{M}^*$ to E^* defined by $[A(\varphi)](x)=\varphi(P(x))$ for $x\epsilon E$, $\varphi\epsilon\mathfrak{M}^*$, are derived. If A is an operation on $\mathfrak{M}^*$ to E^* with the property $[A(\varphi)](x)=\varphi(x)$ for $x\epsilon\mathfrak{M}$, $\varphi\epsilon\mathfrak{M}^*$, then $Q=AT$ is a projection of E^* on $A(\mathfrak{M}^*)$; A sets up an isomorphism between $\mathfrak{M}^*$ and $A(\mathfrak{M}^*)$; if A^{-1} is the inverse of A, then $||A^{-1}||^{-1}\leqq||Q||\leqq||A||$; and if $\mathfrak{M}$ is reflexive then there exists a projection P of E on $\mathfrak{M}$ with the properties (1) $P^*=Q$ and (2) $[A(\varphi)](x)=\varphi(P(x))$ and $||A||=||P||=||Q||$.

A space E is said to have property A if to each closed linear manifold $\mathfrak{M}\subset E$ and $\varphi\epsilon\mathfrak{M}^*$ corresponds a unique $f\epsilon E^*$ such that $f(x)=\varphi(x)$ for $x\epsilon\mathfrak{M}$ and $||f||=||\varphi||$; and to have property B if for each $x_0\neq 0$ in E there is a unique $f\epsilon E^*$ such that $||f||=1$ and $f(x_0)=||x_0||$. Property A implies property B. If E has property B and each closed linear sub-manifold $\mathfrak{M}$ of E is reflexive, then E has property A. If E^* has property B, the unit sphere in E is strictly convex. If the unit sphere in E^* is strictly convex, then E has property A; and if E is reflexive the converse is true. Pertinent examples are given. *R. P. Agnew.*

Lorch, E. R. Bicontinuous linear transformations in certain vector spaces. Bull. Amer. Math. Soc. 45, 564–569 (1939). [MF 52]

Let M be an arbitrary class of elements α and suppose ϕ_α is a function on M to a complex Banach space B. The author calls ϕ_α a basis for B if (*) $0<\inf_M||\phi_\alpha||\leqq\sup_M||\phi_\alpha||<\infty$, and if to each $f\epsilon B$ there belongs a unique complex function a_α satisfying the two conditions that $a_\alpha=0$ except on an at most denumerable set and $\sum_M a_\alpha\phi_\alpha$ converges to f independently of the summation order. Letting $\{\phi_\alpha, \alpha\epsilon M'\}$ be the values taken by ϕ_α on a subset M' of M, the set $\{\phi_\alpha, \alpha\epsilon M\}$ is heterogonal in B if (*) holds and for each $M'\subset M$ every $f\epsilon B$ is uniquely expressible as $f=g+h$, where g is in the span of $\{\phi_\alpha, \alpha\epsilon M'\}$ and h in the span of $\{\phi_\alpha, \alpha\epsilon M-M'\}$. The author shows that (A) ϕ_α is a basis if and only if $\{\phi_\alpha, \alpha\epsilon M\}$ is heterogonal, and (B) if ϕ_α is a basis in a complex Euclidean space B, then ψ_α is a basis if and only if there is a linear bicontinuous transformation T of B onto B such that $T(\phi_\alpha)=T(\psi_\alpha)$ holds for all α.

B. J. Pettis (Cambridge, Mass.).

Paxson, E. W. Linear topological groups. Ann. of Math. 40, 575–580 (1939). [MF 97]

This is a proof and discussion of the following: Every Abelian, convex, connected, and sequentially complete topological (Hausdorff) group, which possesses no elements of finite order, is a pseudo-normed linear topological space. Definitions are, essentially, as in J. v. Neumann [Trans. Amer. Math. Soc. 37, 1–20 (1935)]. *L. Zippin.*

Hyers, D. H. Pseudo-normed linear spaces and abelian groups. Duke Math. J. 5, 628–634 (1939). [MF 178]

Let L be a linear space and D a set of elements satisfying the Moore-Smith postulates for partial order. The author defines a pseudo-norm as a real-valued function $n(x,d)$ on $x\epsilon L$ and $d\epsilon D$ satisfying (1) $n(x,d)\geqq 0$, $n(x,d)=0$ for all $d\epsilon D$ implies $x=\theta$, where θ represents the zero of L, (2) $n(\alpha x, d)=|\alpha|n(x,d)$ for real α, (3) given $\eta>0$, $e\epsilon D$, there exist $\delta>0$, $d\epsilon D$ such that $n(x+y,d)<\eta$ when $n(x,d)<\delta$ and $n(y,d)<\delta$, (4) $n(x,d)\geqq n(x,e)$ for $d>e$. It is proved that the class of pseudo-normable linear spaces is the class of linear topological spaces satisfying essentially von Neumann's postulates [Trans. Amer. Math. Soc. 37, 4 (1935)]; further, that if (3) be replaced by the "triangle" condition (3a) $n(x+y,d)\leqq n(x,d)+n(y,d)$ the class of such spaces is the class of locally convex linear topological spaces. The notion of pseudo-norm is introduced for abelian groups, and it is shown that any pseudo-normable abelian group is continuously isomorphic with a subgroup of a linear topological space. *J. V. Wehausen* (New York, N. Y.).

Moskovitz, David and Dines, L. L. Convexity in a linear space with an inner product. Duke Math. J. 5, 520–534 (1939). [MF 170]

It is shown that in a Banach space in which the norm $||x||=(x,x)^{1/2}$ is given in terms of a real symmetric bilinear form (x,y) every closed convex body is completely supported at its boundary. Also every closed convex set is

supported at a set dense in the boundary. Conversely, every closed set possessing inner points which is supported at a set dense in the boundary is convex. *N. Dunford.*

Kober, H. A theorem on Banach spaces. Compositio Math. 7, 135–140 (1939). [MF 373]

Let E_1+E_2 denote the smallest linear subspace which contains two linearly independent, linear and closed subspaces of a Banach space. The author shows that E_1+E_2 is closed if and only if there is a constant A such that $\|\phi_1\|\leqq A\|\phi_1+\phi_2\|$ for all ϕ_1 in E_1 and ϕ_2 in E_2. The constant A is obviously not less than 1. The author shows that in a Hilbert space the best value for A is 1 if and only if E_1 and E_2 are orthogonal, and in $L_p(a, b)$ the best value for A is 1 if

$$\int_a^b |\phi_1(t)|^{p-2}\phi_1(t)\overline{\phi_2(t)}dt=0$$

for all ϕ_1 in E_1 and ϕ_2 in $E_2(1\leqq p<\infty)$. Applications of these results are made. *I. Halperin* (Kingston, Ont.).

Kakutani, Shizuo. Weak topology and regularity of Banach spaces. Proc. Imp. Acad., Tokyo **15**, 169–173 (1939). [MF 538]

The author gives a new proof of the mean ergodic theorem for uniformly bounded linear operators on uniformly convex Banach spaces. The argument depends on the fact that every uniformly convex Banach space is regular [proved elsewhere by Milman and Pettis], while every regular Banach space is weakly compact. The mean ergodic theorem is a corollary of this and earlier results of Yosida and the author [Proc. Imp. Acad., Tokyo **14**, 292–294 (1938)]. *G. Birkhoff* (Cambridge, Mass.).

Oldenburger, Rufus. Exponent trajectories in symbolic dynamics. Trans. Amer. Math. Soc. **46**, 453–466 (1939). [MF 475]

A symbolic trajectory is a sequence $\cdots a_{-1}a_0a_1a_2\cdots$, where the symbol a_i is chosen from a finite or infinite set of generating symbols. In general T can be considered as a sequence of finite blocks, where the symbols in any one block are the same and the symbols in adjacent blocks are different. The lengths of (number of symbols in) these blocks then form a sequence of integers T_e which is called the exponent trajectory of T. The paper derives a number of relations between T and T_e. If T is periodic or recurrent, T_e is periodic or recurrent, respectively. The converse statements do not hold in general but do hold if T has just two generating symbols. A considerable part of the paper is devoted to a proof that there is one and only one trajectory T in generating symbols 1 and 2 which is identical with its exponent trajectory. *G. A. Hedlund.*

Hartman, Philip and Wintner, Aurel. Asymptotic distributions and the ergodic theorem. Amer. J. Math. **61**, 977–984 (1939). [MF 294]

Let V be a compact, locally Euclidean, n-dimensional manifold of class C^1 such that if μ denotes (the Euclidean choice of) Lebesgue measure on V, then $0<\mu(V)<+\infty$. Let T_t denote for each real t a topological, measure preserving transformation of V into itself such that T_tP (P any point of V) is continuous in the product space $V\times(-\infty<t<+\infty)$. A distribution function $\phi(E)$ on V is defined to be a nonnegative, absolutely additive set function which is defined for every Borel set E in V and is such that $\phi(V)=1$. A Borel set E is a continuity set of ϕ if $\phi(E_*)=\phi(E^*)$, where E_* is the interior and E^* is the closure of E. The set of distribution functions ϕ_w, $0<w<+\infty$, tends to a distribution function ϕ as $w\to\infty$, if $\phi_w(E)\to\phi(E)$ holds for every fixed continuity set E of ϕ. Let $[P]$ denote the set P_t, $-\infty<t<+\infty$; let $f(P, E)$ be the characteristic function of the Borel set E; and let

$$\phi_{uv}^P(E)=(v-u)^{-1}\int_u^v f(P_t, E)dt.$$

The path $[P]$ is said to possess an asymptotic distribution function ϕ^P if ϕ_{uv}^P tends, as $v-u\to\infty$, to the distribution function ϕ^P. The Birkhoff ergodic theorem implies and is implied (for the system under consideration) by the following theorem: The asymptotic distribution function ϕ^P of the path $[P]$ exists whenever P does not belong to a certain zero set Z_0 of V (where Z_0 is uniquely determined by the system).

The spectrum of a distribution function $\phi(E)$ is the set of those points Q of V such that $\phi(G)>0$ whenever G is an open set containing Q. The main result of the paper is the following simultaneous refinement of the ergodic theorem and the Poincaré recurrence theorem. The asymptotic distribution function ϕ^P of the path $[P]$ exists and has the closure $[P]'$ of $[P]$ as spectrum whenever P does not belong to a certain zero set Z of V. *G. A. Hedlund.*

Kakutani, Shizuo. Mean ergodic theorem in abstract (L)-spaces. Proc. Imp. Acad., Tokyo **15**, 121–123 (1939). [MF 537]

By an abstract (L)-space is meant a space having various linear, metric, and lattice-theoretic properties common to the space (L), the space (l), and the space (V). F. Riesz has proved a mean ergodic theorem for elements whose transforms are lattice-theoretically bounded, under a cyclic semi-group of linear operators of bounded modulus, on the space (L). [Cf. J. London Math. Soc. **13**, 274–278 (1938).] The author shows that this theorem holds in any abstract (L)-space, verifying a conjecture of the reviewer. This result is important in the theory of dependent probabilities. The main point is the possibility of embedding a separable abstract (L)-space isometrically and lattice-isomorphically in the concrete space (L), proved by Freudenthal [Nederl. Akad. Wetensch., Proc. **39**, 641–651 (1936)]. *G. Birkhoff* (Cambridge, Mass.).

Yosida, Kôsaku and Kakutani, Shizuo. Birkhoff's ergodic theorem and the maximal ergodic theorem. Proc. Imp. Acad., Tokyo **15**, 165–168 (1939). [MF 536]

Let S be a space in which a measure of Lebesgue type is defined; it is not assumed that meas (S) is finite. Let $f(x)$ be a real valued function defined on S, and T a one-to-one measure preserving transformation of S into itself. Let

$$f^*(x)=\sup_n \frac{1}{n}\sum_{i=0}^{n-1}f(T^ix), \qquad f_*(x)=\inf_n \frac{1}{n}\sum_{i=0}^{n-1}f(T^ix),$$

$$E^*(\alpha)=\underset{x}{E}[f^*(x)>\alpha], \qquad E_*(\alpha)=\underset{x}{E}[f_*(x)<\alpha].$$

Then, if $f(x)$ is measurable and integrable on S, we have

$$\alpha \text{ meas } (E^*(\alpha))\leqq\int_{E^*(\alpha)} f(x)dx,$$

$$\alpha \text{ meas } (E_*(\alpha))\geqq\int_{E_*(\alpha)} f(x)dx.$$

The authors give a simple direct proof of this theorem

which is a generalization of a recent theorem by N. Wiener [Duke Math. J. **5**, 1–18 (1939), Theorem IV], which was proved by an entirely different method. The method used by the authors is a modification of the method used by Kolmogoroff in proving the classical ergodic theorem of G. D. Birkhoff. *J. D. Tamarkin* (Providence, R. I.).

Celestial Mechanics, Relativity

Pedersen, P. Über eine Klasse infinitesimaler, periodischer Bahnen um die Dreieckslibrationspunkte im problème restreint. Astr. Nachr. **269**, 31–40 (1939). [MF 7]

The above named paper is based on a previous paper by the author [Monthly Not. Roy. Astr. Soc. **95**, 482]. Both papers treat the restricted problem of three bodies and consider only plane periodic orbits of the infinitesimal body near the equilateral triangle points. In the original paper the author reports that he has shown the existence of two types of orbits, one a long period libration ellipse and the other a short period ellipse. Only terms of the third degree in the series development of the right members of the differential equations were considered. A special case arose in which it was not possible to compute the coefficients of the series for the long period orbits. This case occurred when the rotation time of one class of orbits was exactly twice that of the other. The present paper considers the special case in which the rotation time of the long period orbits differs from twice the rotation time of the short period orbits by an infinitesimal of the first order. It is shown that the long period orbits form a closed class of infinitesimal orbits with an infinitesimal outer boundary orbit. As the rotation time of the long period orbits approaches a value which is double that of the short period orbits the outer boundary orbit reduces to the libration point and the long period orbits do not exist. *H. E. Buchanan.*

Pedersen, Peder. Fourier expansions for periodic orbits around the triangular libration points. Danske Vid. Selsk. Math.-Fys. Medd. **17**, no. 4, 16 pp. (1939). [MF 440]

The paper supplements two earlier papers [Monthly Not. Roy. Astr. Soc. **95**, 482 (1934–35)] in finding periodic orbits around the triangular libration points in the restricted problem of three bodies. Here the author has carried out the complete calculation of the Fourier series up to terms of the third order, for a special case which his previous work did not cover, where the product of the masses is approximately 1/27. *E. J. Moulton* (Evanston, Ill.).

Baños, Alfredo, Jr. On asymptotic orbits in the theory of primary cosmic radiation. J. Math. Phys. Mass. Inst. Tech. **18**, 211–238 (1939). [MF 193]

It has been shown by Lemaitre and Vallarta that primary charged cosmic rays, of a given energy, moving in the earth's magnetic field, can reach a point on the earth's surface only along certain allowed directions. The latter fill a cone of many sheets called the allowed cone. The generators of the cone are in part orbits asymptotic to a certain family of unstable periodic orbits, which have the property of separating trajectories which go to infinity without having maxima and minima from those which do not.

The author gives a calculation of a family of asymptotic orbits corresponding to the value $\gamma_1=0.85$ of Störmer's parameter. This calculation falls into three parts: first, the outer principal periodic orbit belonging to γ_1 is determined, using Lemaitre's method; second, an asymptotic expansion is found in terms of Poincaré's characteristic exponents, valid in the vicinity of the periodic orbit; and third, the asymptotic expansion is continued by numerical integration. For this purpose the family of asymptotic orbits is represented by trigonometric series of the argument $\omega\sigma+\varphi$ (where ω is the circular frequency of the periodic motion, σ the independent variable depending on the time, and φ an arbitrary phase angle serving to single out individual members of the asymptotic family) and the amplitudes of the different components, only a limited number of which are appreciable, are determined by numerical integration, adapting a method devised by Lemaitre and Vallarta. The results are compared with those obtained by these two authors using the method of mechanical integration, and the agreement is found to be quite satisfactory. A symmetric doubly asymptotic orbit is calculated and it is surmised that, except for a set of zero measure, all orbits belonging to a family asymptotic in the past go to infinity as time increases without limit. *M. S. Vallarta.*

Fabre, Hervé. Librations des apsides de certaines orbites peu excentriques. C. R. Acad. Sci. Paris **209**, 151–153 (1939). [MF 227]

Vescan, Théophile T. Sur les orbites relativistes des planètes. C. R. Acad. Sci. Paris **209**, 149–151 (1939). [MF 226]

Kopal, Zdeněk. The dynamics of double star systems and stellar density condensations. Z. Astrophys. **18**, 272–283 (1939). [MF 77]

The paper deals with the dynamics of close binary systems, the components of which can be regarded as invariable in form. It is shown that such bodies describe orbits which, if their eccentricities are not too large, exhibit an advance of the longitude of periastron of a given amount. The ellipsoidal components are free to display in the orbital plane two modes of librations. The amplitudes of the short-periodic librations (their periods being slightly different from that of the orbit) are found to be proportional to the orbital eccentricity and vanish for circular orbits. *Extract from the paper.*

Chandrasekhar, S. The dynamics of stellar systems. I–VIII. Astrophys. J. **90**, 1–154 (1939). [MF 275]

The problem is that of the dynamics of a stellar system, such as the galaxy, in which the individual stars in any small region may be considered as moving with respect to a local centroid, which serves as a standard of rest for that region, and which is in motion with respect to the other local centroids. Without making use of the previous scattered attempts at this problem, the author gives it a definite formulation, and initiates a determined attack on its solution. He assumes that the dependence of the distribution function on the velocity, relative to the local centroids, is defined by a homogeneous quadratic form, and that the entire system is in a stationary state under the influence of a given potential. The general equations for such a system are obtained in Part I; they show that the problem is in fact reducible to that of determining the most general integral, of at most second degree in the components of velocity, of the equations of motion of a single particle in the given potential field. Certain of these equations do not involve the potential function, and in Part III the necessary conditions are solved for the coefficients of the quadratic

and linear terms in the distribution functions; they are found to depend linearly on at most twenty and six arbitrary constants, respectively. In Part IV it is shown that this latter result leads, for any finite system whose local centroids exhibit differential motions, to the result that the potential must have a spatial symmetry defined by a Euclidean motion – a translation or a screw. The remaining sections are devoted to the recovery of these results in various orthogonal coordinate systems, and to the solution of special problems by imposing conditions on the potential; thus in Part II the general 2-dimensional problem is solved, in Part V the general spherically symmetric case, and in Parts VI–VIII certain cases most naturally treated in spheroidal, cylindrical and ellipsoidal coordinates.

H. P. Robertson (Princeton, N. J.).

Einstein, Albert. On a stationary system with spherical symmetry consisting of many gravitating masses. Ann. of Math. **40**, 922–936 (1939). [MF 331]

The author investigates the general relativistic problem of the field of a stationary spherically symmetric distribution of particles, each of which moves in a circular orbit under the gravitational influence of the swarm. On inserting the matter-energy tensor of such a statistical distribution into the field equations, he finds that the radial distribution of particles, although functionally arbitrary, is subject to a numerical restriction of such a nature that the coefficients of the line-element can nowhere have a singularity such as that encountered at the point $r=2m$ of the Schwarzschild solution. The author conjectures that this behaviour will be exhibited in all cases, and concludes that "the 'Schwarzschild singularity' does not appear for the reason that matter cannot be concentrated arbitrarily . . . due to the fact that otherwise the constituting particles would reach the velocity of light." *H. P. Robertson.*

Lichnerowicz, André. Nouvelles recherches sur les singularités des espaces-temps extérieurs. C. R. Acad. Sci. Paris **209**, 533–534 (1939). [MF 523]

In this note the author has shown how a result dealing with a distribution of matter in the theory of relativity which was previously established in his thesis is likewise obtained when this distribution is replaced by a distribution of electromagnetic energy. [See his thesis: Sur certains problèmes globaux relatifs au système des équations d'Einstein, Paris, 1939, pp. 65–69.] *T. Y. Thomas.*

Armellini, G. I problemi fondamentali della cosmogonia e la legge di Newton. IV. Atti Accad. Naz. Lincei. Rend. **29**, 649–655 (1939). [MF 514]

The fourth of a series of notes on the cosmogonic consequences of modifying the Newtonian law of attraction by multiplying the force by $1+\epsilon dr/dt$, where ϵ is a small positive constant. It was shown in the previous notes that this resistance term causes the eccentricity of a planet to decrease, and the plane of the orbit to approach that of the rotating sun's equator, in such a way as to give rise to a rotation in this plane in the same sense as that of the sun about its axis. The present note evaluates an upper limit for the constant ϵ, by requiring that the effect on the mean longitude of the earth be consistent with the observation; the value $\epsilon=4\cdot10^{-14}$ (c.g.s. units) thus obtained is shown to give rise to secular changes in the eccentricity and inclination of the orbit which, although below the limits of direct observation, are of possible cosmogonic significance.

H. P. Robertson (Princeton, N. J.).

Theory of Probability

✶von Mises, Richard. Probability, Statistics and Truth. William Hodge and Co., Ltd., London, 1939. xvi+323 pp. 12/6.

A popular non-mathematical account of the author's well-known theory of probability. Lecture I: the scientist's precision of popular terms in general, and the author's definition of probability and "randomness" in particular. Lecture II: the elements of the von Mises theory, its definitions and fundamental operations. Lecture III: a critical study in which the well-known defects of classical theories of the foundations of probability are reviewed, and criticisms of the von Mises theory are answered. Lecture IV: the law of large numbers from the point of view of the present theory. Lecture V: its application to statistics and the theory of errors. Lecture VI: contains a popularized account of statistical mechanics, quantum mechanics, and causality, with particular reference to the author's theory.

To the technical reader, the essentially novel parts of the book are the discussion of criticisms of the von Mises theory in Lecture III. The author seeks to answer the well-known objections to his definition of "randomness" (Regellosigkeit) by an application of recent results of Copeland and Wald; yet this application appears to be based on a fallacious use of the notion of class of all definable "place selections" [p. 144]. The author considers also the familiar difficulty with all "objective" definitions of probability as a density in a sequence (that is, a limit). The finite number of observations which I can ever make on individual terms of the sequence has no logical connection with the existence or value of the limit. This is answered by analogy with the case of specific gravity [p. 124], itself the lim (weight/volume) as volume approaches 0, and can only be measured by weighing a finite volume. But we find all such analogies unconvincing, inasmuch as in the latter case the experimenter is permitted to draw "very convincing" or "practically certain" conclusions as to the density from his measurements on finite volumes, that is, to appeal to the "subjective" notion of probability, whereas this is not allowed in the sequence definition of probability from which all vestiges of "subjective probability" are banished. If finite collectives are used [p. 121], they are applied either to measurements on the majority of their terms, in which case the theory is useless, or else on measurements on minute subsets of their terms, whereupon all the above difficulties reappear.

B. O. Koopman (New York, N. Y.).

McKinsey, J. C. C. A note on Reichenbach's axioms for probability implication. Bull. Amer. Math. Soc. **45**, 799–800 (1939). [MF 346]

Attention is drawn to a contradiction and a desirable modification in the axioms given by H. Reichenbach [Wahrscheinlichkeitslehre, Leiden, 1935] as foundations of the theory of probability and addition for a system of logic.

W. Feller (Providence, R. I.).

Greville, T. N. E. Invariance of the admissibility of numbers under certain general types of transformations. Trans. Amer. Math. Soc. **46**, 410–425 (1939). [MF 472]

The author considers Copeland's admissible numbers, which are to satisfy certain further properties, besides those Copeland required. It is shown that there are "numbers" with these properties (that is, sequences of zeros or ones), essentially by showing that these properties, like Copeland's, are true with probability 1, that is, that almost all

sequences have these properties. The author's division of the problem into two cases, according as the given probability is rational or not, is unnecessary, since a probability measure can be defined on these "numbers" without mapping them on a line segment. *J. L. Doob* (Urbana, Ill.).

Fréchet, Maurice. A note on the "problème des rencontres." Amer. Math. Monthly **46**, 501 (1939). [MF 363]

Remark concerning a paper by I. Kaplansky [Amer. Math. Monthly **46**, 159–161 (1939)].

Geiringer, Hilda. Bemerkung zur Wahrscheinlichkeit nicht unabhängiger Ereignisse. Rev. Math. Union Interbalkan. **2**, 1–7 (1939). [MF 592]

The author considers the relation between systems of events in which the probabilities of the various conjunctions are given and systems in which events are thought of as occurring in sequence and the various conditional probabilities are given. Certain special cases are treated.

A. H. Copeland (Ann Arbor, Mich.).

Kac, M. On a characterization of the normal distribution. Amer. J. Math. **61**, 726–728 (1939). [MF 31]

Let $f(\alpha)$ and $g(\alpha)$ be (real) measurable functions defined in $(0, 1)$ which are statistically independent and such that $f(\alpha)+g(\alpha)$ and $f(\alpha)-g(\alpha)$ are also statistically independent. The author proves that the distribution functions $\sigma_f(\omega)=\text{meas}\,[f(\alpha)<\omega]$ and $\sigma_g(\omega)=\text{meas}\,[g(\alpha)<\omega]$ of $f(\alpha)$ and $g(\alpha)$ are then normal and symmetric with the same precision. The independence leads to functional equations for the Fourier transforms of $\sigma_f(\omega)$ and $\sigma_g(\omega)$, which in case of symmetry are solved in the Cauchy manner; the general case is reduced to the case of symmetry by means of a theorem of Cramér. The extension of the theorem to asymptotic distribution functions is immediate. A consequence of the theorem is that, if the components of an n-dimensional vector function are statistically independent in every rectangular system of coordinates, then its distribution function is Gaussian and of radial symmetry.

B. Jessen (Copenhagen).

Lévy, Paul. L'addition des variables aléatoires définies sur un circonférence. Bull. Soc. Math. France **67**, 1–41 (1939). [MF 443]

In this paper, Lévy develops earlier results [C. R. Acad. Sci. Paris **207**, 444–446 (1938)] further, and gives their detailed proofs. He discusses chance variables defined on the perimeter Γ of a circle of radius $1/2\pi$, that is, defined modulo 1. Many of the results hold when the distribution functions are of bounded variation without being monotone non-decreasing, or even if the total mass is infinite. He finds the stable distributions (a distribution is stable if the sum of two independent chance variables with the given distribution also has the given distribution). In the probability case, for example, a stable distribution is necessarily uniform, either over the whole perimeter or on the vertices of a regular inscribed polygon. A generalized expectation $\mathrm{E}x$ and dispersion $\sigma^2(x)$ of the chance variable x are defined, and it is shown that, if $x_1, x_2, \cdots$ are mutually independent, $\sum_j(x_j-\mathrm{E}x_j)$ converges (with probability 1) if and only if $\sum_j\sigma^2(x_j)<\infty$. Random functions x_t are then discussed, for which, if $t_0<\cdots<t_n$, $x_{t_1}-x_{t_0}, \cdots, x_{t_n}-x_{t_{n-1}}$ are independent. There is a certain closed exceptional t-set, described in the paper, outside of which Lévy's well-known results [Ann. Scuola Norm. Super. Pisa (2) **3**, 337–366 (1934); **4**, 217–218 (1935)] on such random functions defined on a line instead of on Γ, go over: $x_t-\mathrm{E}x_t$ can be assumed continuous except for jumps, etc. *J. L. Doob.*

Lévy, Paul. Extensions stochastiques des notions de série, d'intégrale et d'aire. C. R. Acad. Sci. Paris **209**, 591–593 (1939). [MF 519]

Let $\sum u_h$ be a conditionally convergent series and let α_h be a sequence of positive numbers such that $\alpha_h\to 0$ as $h\to\infty$. In case $\sum\alpha_h<\infty$ one takes at random one of the u's as the first term U_1 of a new series (the probability of drawing u_h is proportional to α_h); one removes then the chosen term and one proceeds in the same way to define U_2 and so on. In case $\sum\alpha_h=\infty$ one should remove besides the chosen terms also all the terms whose corresponding "weights" are less than a certain $\bar\alpha_n$, specified by the author in a rather arbitrary way. The author announces now (without indication of a proof) the following results: (1) It is "almost certain" (in the probabilistic sense) that the rearranged series $\sum U_h$ will contain all the u's. (2) There exist two numbers s' and s'' (finite or infinite) so that the relations

$$\liminf_{n\to\infty}\,(U_1+\cdots+U_n)=s', \quad \limsup_{n\to\infty}\,(U_1+\cdots+U_n)=s''$$

hold with the probability 1.

If $s'=s''=s$ and s is finite, $\sum u_h$ is called "stochastically convergent" with respect to the "weights" α_h and s is called the "stochastic sum." For the series $\sum\pm 1/h$ and $\alpha_h=[1-(-1)^hc]/h$ $(|c|<1)$ the author gets $s=\log 2+\frac{1}{2}\log((1+c)/(1-c))$. The "stochastic integral" and in particular the "stochastic length" and the "stochastic area" are defined in a similar way. *M. Kac* (Ithaca, N. Y.)

Halmos, Paul R. On a necessary condition for the strong law of large numbers. Ann. of Math. **40**, 800–804 (1939). [MF 323]

Let x_n be a sequence of mutually independent random variables and write $\bar x_n=x_n$ whenever $|x_n|\leqq n$, and $\bar x_n=0$ otherwise. Put $b_n=E([\bar x_n-E(\bar x_n)]^2)$. Suppose that (i) $\lim\,\{E(x_1)+\cdots+E(x_n)\}/n=0$, (ii) $P\{\lim x_n/n=0\}=1$, and (iii) that $\sum b_n/n^2$ converges. Then the strong law of large numbers holds for x_n, that is to say, (iv) $P\{\lim\,(x_1+\cdots+x_n)/n=0\}=1$. This is stated by the author to be a slight generalization of a well-known theorem of Kolmogoroff [Grundbegriffe der Wahrscheinlichkeitsrechnung, Berlin, 1933, p. 59]. The purpose of the paper is the proof of the following converse: The relations (i) and (iv) together imply (ii) and the convergence of $\sum b_n/n^{2+\epsilon}$ for all $\epsilon>0$. (It may be remarked that the convergence of $\sum b_n/n^{2+\epsilon}$ is necessary even for the weak law of large numbers; this is an immediate consequence of the necessary and sufficient conditions given by the reviewer, Acta Litt. Sci. Szeged **8** (1937); alternative proof by Marcinkiewicz, Fund. Math. **30** (1938).)

W. Feller (Providence, R. I.).

Yosida, K. and Kakutani, S. Markoff process with an enumerable infinite number of possible states. Jap. J. Math. **16**, 47–55 (1939). [MF 531]

Elegant new proofs are given that if $p_{ij}^{(n)}$ is the probability of going from the ith to the jth state in n steps of a Markoff process $(i, j=1, 2, \cdots)$, then

$$\lim_{N\to\infty}\,(1/N)\sum_{n=1}^{N}p_{ij}^{(n)}$$

exists for all i, j, and the collection of states can be separated into a dissipative part and ergodic parts.

J. L. Doob (Urbana, Ill.).

van Kampen, E. R. and Wintner, Aurel. A limit theorem for probability distributions on lattices. Amer. J. Math. 61, 965–973 (1939). [MF 292]

The authors discuss the random walk problem on a k-dimensional discrete lattice; it is supposed that a point moves on this lattice, the movement being determined by a given distribution function whose spectrum is in the lattice. This problem was discussed by Pólya [Math. Ann. 84, 149–160 (1921)] in the case of a cubic lattice in which each step takes place in accordance with a symmetric Bernoulli distribution. The authors suppose only that the expected value of the motion is 0, and that the scalar second moment is neither 0 nor $+\infty$. It is shown that the spectrum of the distribution function tends towards periodicity in the following sense (made numerically precise in the paper). The lattice of possible positions can be divided into subsets $L_1, \cdots, L_r$ such that the distribution after $rm+h$ steps $(0 \leqq h < r)$ becomes (as $m \to \infty$) an equidistribution over any given bounded portion of L_h, and just as in the Bernoulli case, the portion can be varied with m so that the k-dimensional normal distribution appears as the limit distribution in L_h. *J. L. Doob* (Urbana, Ill.).

Haviland, E. K. Asymptotic probability distributions and harmonic curves. Amer. J. Math. 61, 947–954 (1939). [MF 290]

The general theory of addition of independent distributions on convex curves [cf. E. R. van Kampen and A. Wintner: Convolutions of distributions on convex curves and the Riemann zeta function, Amer. J. Math. 59, 175–204 (1937), where further references are given] is applied to the particular case of two ellipses. The results thus obtained are applied to the investigation of some experimental results concerning so-called harmonic curves. *M. Kac* (Ithaca, N. Y.).

Theoretical Statistics

✱Treloar, Alan E. Elements of Statistical Reasoning. John Wiley & Sons, Inc., New York, 1939. xi+261 pp. $3.25.

An introductory text for graduate students of science, dealing with many traditional fundamental statistical concepts, using simple language and with effective diagrams. The book is devoid of exercises for home study, has few historical references save to J. Arthur Harris, and does not mention index numbers, seasonal variation, or other topics dear to economic statisticians. "The author believes that unfortunate consequences flow all too readily from an approach to statistical reasoning via 'the small sample.'" The critical analysis of small-sample techniques is excluded for lack of space. *A. A. Bennett* (Providence, R. I.).

✱Rider, Paul R. An Introduction to Modern Statistical Methods. John Wiley & Sons, Inc., New York, 1939. ix+220 pp. $2.75.

This book endeavors to explain the routine application of some modern statistical methods, mainly by numerical illustrations. It is chiefly devoted to the theory of testing statistical hypotheses, the discussion being entirely based on R. A. Fisher's work. Having discussed the fundamental notions and properties of frequency distributions, averages and moments, regression and correlation, the author explains the methods of testing the significance of the correlation coefficient or the difference between such coefficients, or of the significance between two means. The χ^2 and Student's distributions are introduced together with their applications. The book concludes with chapters devoted to the analysis of variance and the experimental design.

However, the author makes no attempt toward giving any explanations or a theory in a mathematical sense. Thus no attention is paid to the difference between theoretical and empirical distributions; the fundamental conceptions are described rather vaguely and the reader will find confusing definitions such as [p. 88]: "Any quantity such as a standard deviation, a median, a correlation coefficient, when calculated from a sample, is called a statistic; the corresponding quantity in the population is called a parameter." *W. Feller* (Providence, R. I.).

Delaporte, Pierre. Une méthode d'analyse des corrélations et son application. C. R. Acad. Sci. Paris 209, 142–145 (1939). [MF 223]

A new method for factor analysis is discussed. Instead of the tetrad differences, confidence intervals (intervalles d'échantillonnage) of the ratios r_{AJ}/r_{BJ} are considered, where A, B, J denote some characters and r_{AJ} and r_{BJ} denote the correlation coefficients between A and J and B and J, respectively. The symbol AB $(cd \cdots l)$ is introduced to denote the relationship that the confidence intervals of the ratios $r_{A\Gamma}/r_{B\Gamma}$ for $\Gamma = C, D, \cdots, L$ have a common part. The symbol AB $(cd, efg, h \cdots l)$ denotes the relationship that there are three different common parts, one for CD, one for EFG, and one for $H \cdots L$. These and analogous relationships among the confidence intervals of the ratios r_{AJ}/r_{BJ} are used for determining the general factor and different group factors. *A. Wald.*

Pitman, E. J. G. A note on normal correlation. Biometrika 31, 9–12 (1939). [MF 256]

Using the familiar fact that two simple linear combinations of two normally correlated variates are independently and normally distributed, an exact test is derived for the significance of the ratio of sample variances in samples from a normal bivariate population. The test contains only quantities calculable from the sample and can be put in the form of a Fisher's "t." From this fiducial limits for the ratio of population variances are found. There is a numerical example and special cases are discussed. *C. C. Craig.*

Pitman, E. J. G. Tests of hypotheses concerning location and scale parameters. Biometrika 31, 200–215 (1939). [MF 263]

Suppose that we are given a sample of k random variables, the simultaneous frequency distribution of which is either $F(x_1-a_1, \cdots, x_k-a_k)$ or $F(x_1/a_1, \cdots, x_k/a_k)/(a_1 \cdots a_k)$, where F is a known continuous function, the values of the a_i being unknown. The author first derives tests for the statistical hypothesis $a_1 = \cdots = a_k$, which are unbiased in the sense of Neyman and E. S. Pearson and are based on the statistics

$$\int_{-\infty}^{\infty} F(x_1-a, \cdots, x_k-a)\,da$$

or

$$\frac{1}{(x_1 \cdots x_k)} \int_0^{\infty} F(x_1/a, \cdots, x_k/a)\,da/a^{k+1},$$

respectively. The hypothesis $a_1 = \cdots = a_k = \alpha$, where α is a given value, is also treated. The general results are obtained

by an elementary argument. The properties and the distribution of the tests are studied in detail in the special case where F is the product of k functions $e^{-x_r/a_r}x^{m_r-1}/a_r^{m_r}\Gamma(m_r)$ with $x>0$ and known m_r. The most important application is that to the comparison of variances of normally distributed variables. The author's method yields Bartlett's test [Proc. Roy. Soc. London Ser. A. **160**, 268–282 (1937)], whereas the original L_1 test by Neyman and Pearson [Bull. Int. Acad. Polon. Sci. A. **1931**, 460–481, and Statist. Res. Mem. London **1**, 1–37 (1936)] is shown to be somewhat biased except when the samples are all of the same size.

W. Feller (Providence, R. I.).

Morgan, W. A. A test for the significance of the difference between the two variances in a sample from a normal bivariate population. Biometrika **31**, 13–19 (1939). [MF 257]

Let x and y denote two random variables known to be normally distributed about unknown means, with unknown standard deviations σ_1 and σ_2, and with unknown correlation coefficient ρ. The observation may provide n pairs of corresponding values of x and y. The author deduces the Neyman-Pearson λ criterion to test the hypothesis that $\sigma_1=\sigma_2$. If $x+y=2u$ and $x-y=2v$, and if r denotes the correlation coefficient between the observed values of u and v, then it appears that $\lambda=(1-r^2)^{n/2}$ and therefore the probability law of λ, both on the hypothesis tested and on any of the alternatives, is easily obtained from that of r. This connection makes it easy not only to establish all the details of the test but also to calculate the power function. The author shows that the latter depends on the value of ρ and namely that the stronger the correlation between x and y, the greater the power of the test. The new test is being compared with that previously suggested, on intuitive grounds, by Finney [Biometrika **30**, 190 (1938)], which, however, is only applicable when the value of ρ is known.

J. Neyman (Berkeley, Calif.).

Bishop, D. J. On a comprehensive test for the homogeneity of variances and covariances in multivariate problems. Biometrika **31**, 31–55 (1939). [MF 259]

In this paper the author has considered in some detail the application of the likelihood criterion L_1, derived by Wilks [Biometrika **24** (1932)], for testing the hypothesis H_0 that k samples are from normal populations of q variables having the same matrix of variances and covariances. For the case of samples of the same size n, the distribution function of L_1, when H_0 is true, is approximated by fitting a Pearson Type I curve, using the first two moments of L_1. The significance of an observed value of L_1 for a given sample size and a given value of k is then determined from the Tables of the Incomplete Beta Function, or alternately from R. A. Fisher's z-tables. The adequacy of the fit of the Type I curve is studied by comparing the first four moments of the exact distribution of L_1 with those of the fitted distribution for the following combinations of values of k, n, q: (1) $k=5$; $n=10, 20, 40, 50$; $q=1, 2, 3, 4, 5, 6$ and (2) $k=2, 5, 10$; $n=30$; $q=1, 2, 3, 4, 5, 6$. The adequacy of fit is also studied by comparing probabilities from the actual and fitted distributions for the 1% and 5% levels for $k=2$; $q=2, 1$; $n=10, 30, 50$. In case of large samples, it is known that $-2N\log L_1$, where N is the sum of the sample sizes for all samples, is approximately distributed according to the χ^2 law with $\frac{1}{2}(k-1)q(q+1)$ degrees of freedom. A comparison is made of the fits obtained by the Type I curve and by the χ^2 approximation for several combinations of values of q, k, and n.

S. S. Wilks (Princeton, N. J.).

Jeffreys, Harold. The minimum χ'^2 approximation. Proc. Cambridge Philos. Soc. **35**, 520 (1939). [MF 554]

A result of an earlier paper by the author [Proc. Cambridge Philos. Soc. **34**, 156–157 (1938)] has, as he now points out, been anticipated by Neyman [Bull. Inst. Intern. Warsaw **1929**, 44–86].

W. Feller (Providence, R. I.).

Wilks, S. S. and Daly, J. F. An optimum property of confidence regions associated with the likelihood function. Ann. Math. Statistics **10**, 225–235 (1939). [MF 147]

Let x stand for a system of m random variables depending on h parameters $\theta_1, \theta_2, \cdots, \theta_h$, the values of which are unknown. Consider h functions $h_i(x, \theta)$, for $i=1, 2, \cdots, h$, where θ stands for the system of all the parameters θ_j. Suppose that it is possible to make an infinite sequence of independent observations $x_1, x_2, \cdots, x_n, \cdots$ of values of x and consider the sequences $\{h_i(x_n, \theta)\}$ of corresponding values of each of the functions $h_i(x, \theta)$. The considerations are limited to the class K of functions $h_i(x, \theta)$ for which the sequences $\{h_i(x, \theta)\}$ satisfy the conditions of the Liapounoff-Bernstein Theorem, that is, are such that, as $n\to\infty$, the joint probability law of the properly standardized sums

$$H_{i,n}=A_{i,n}\sum_{j=1}^{n} h_i(x_j, \theta), \qquad i=1, 2, \cdots, h,$$

tends to a normal distribution in the h dimensioned space. Denote by $\phi_{i,n}$, for $i=1, 2, \cdots, h$, linear combinations of the $H_{i,n}$ such that the limiting elementary probability law of the $\phi_{i,n}$ is given by

$$f=(2\pi)^{-h/2}\cdot\exp\left[-(1/2)\sum_{i=1}^{h}\phi_i^2\right].$$

The authors consider the problem of the joint estimation of the parameters $\theta_1, \cdots, \theta_h$ and suggest a class of systems of confidence regions constructed as follows. Denote by α the confidence coefficient chosen and by R a number such that the integral of f with respect to $\phi_1, \phi_2, \cdots, \phi_h$ taken over the region S, defined by the inequality $\sum\phi_i^2\leqq R$, is equal to α. Now fix any system X' of values of $x_1, x_2, \cdots, x_n, \cdots$ which may be determined by observation and denote by θ' the system of values of θ for which the point determined by $\phi_{i,n}(X', \theta)$, $i=1, 2, \cdots, h$, falls within S. It is easy to see that, if n is sufficiently large, θ' represents, approximately, a confidence region for $\theta_1, \cdots, \theta_h$ corresponding to the experimental data X' and to the confidence coefficient α. The main result of the paper is that, whenever the probability law of the x's satisfies certain limiting conditions, the derivatives of the logarithm of the likelihood taken with respect to the particular θ_i's belong to the class K and, moreover, the Jacobian with respect to $\theta_1, \theta_2, \cdots, \theta_n$ of $\phi_1, \phi_2, \cdots, \phi_n$, calculated starting with those derivatives of the logarithm of the likelihood treated as $h_i(x, \theta)$, has an expected absolute value which cannot be exceeded by that corresponding to any other system of functions $h_i(x_j, \theta)$ belonging also to class K. The authors express the opinion that the above circumstance has a favorable bearing on the "size" of the confidence regions and propose to call those based on the derivatives of the likelihood the smallest average confidence regions. The paper reviewed is a generalization of the previous interesting results of S. S. Wilks [Ann. Math. Statistics **9**, 166–175 (1938)].

J. Neyman.

AUTHOR INDEX

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